Hurwitz Matrix for Polynomial Matrices
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Abstract. If a system is given by its transfer function then the stability of the system is determined by the denominator polynomial and its corresponding Hurwitz matrix $\mathcal{H}$. Also the critical stability conditions are determined by its determinant $\det(\mathcal{H})$.

The aim of this paper is to get a generalized Hurwitz matrix for polynomial matrices. In order to achieve that, we first obtain a relation between the Hurwitz matrix for a polynomial and the Lyapunov equation. Here we show how the Hurwitz matrix appears in the solution of the Lyapunov equation using the companion matrix realization and the Kronecker formulation of Lyapunov equation. Using this result we show how the generalized Hurwitz matrix for polynomial matrices can be constructed.


For a dynamic system given by the rational transfer function

$$G(s) = \frac{b(s)}{a(s)}$$

with the denominator polynomial

$$a(s) = a_0 + a_1 s + a_2 s^2 + \cdots + a_{n-1} s^{n-1} + s^n$$

the Hurwitz matrix $\mathcal{H} \in \mathbb{R}^{n \times n}$ is given by

$$\mathcal{H} = \begin{bmatrix}
a_{n-1} & a_{n-3} & a_{n-5} & \cdots & 0 & 0 \\
1 & a_{n-2} & a_{n-4} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\cdots & a_{n-1} & a_{n-3} & a_{n-5} & \cdots & 0 \\
\cdots & 1 & a_{n-2} & a_{n-4} & \cdots & 0 \\
\cdots & \cdots & a_3 & a_1 & 0 \\
\cdots & \cdots & a_4 & a_2 & a_0 \\
\end{bmatrix}.$$  

For stability of (1) it is necessary and sufficient that all the leading principal minors of $\mathcal{H}$ are positive. Also the critical stability conditions are determined by

$$\det(\mathcal{H}) \neq 0.$$  

It is well known that Lyapunov theory gives an alternative method to investigate stability of (1). A realisation of (1) is given by

$$\dot{x} = A x + b u$$
where \( a(s) = \det(sI - A) \) and the system matrix \( A \)

\[
A = \begin{bmatrix}
-a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\
1 & 0 & \cdots & \cdots & 0 \\
0 & 1 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 1 & 0
\end{bmatrix}
\]

(5)

is in companion form.

The stability of (1) or (4) is given by the solution of the Lyapunov equation [4]

\[
A'P + PA = -Q.
\]

(6)

The matrix \( A \) is stable if and only if for arbitrary positive definite \( Q > 0 \) there exists a positive definite solution for \( P \) in (6).

In order to obtain the critical stability conditions in this case we consider the Kronecker formulation of (6) [4], [6] :

\[
(A' \otimes I + I \otimes A') \col(P) = -\col(Q).
\]

(7)

The eigenvalues of the operator \( \mathcal{A} \) are given by

\[
\operatorname{Eig}(\mathcal{A}) = \lambda_i + \lambda_j \quad \text{for } \forall i, j
\]

where \( \lambda_k \) are the eigenvalue of \( A \). Therefore

\[
\det \mathcal{A} \neq 0
\]

(8)

gives the critical stability conditions or the guardian map [5] for the stability of \( A \). The matrix \( \mathcal{A} \in \mathbb{R}^{n^2 \times n^2} \) in (7) can be reduced to \( \mathcal{A}_r \in \mathbb{R}^{(n+1)n/2} \) due to symmetry of the matrix \( P \). In the following we shall use the reduced operator \( \mathcal{A}_r \) instead of \( \mathcal{A} \).

We noticed that (3) and (8) are equivalent. However instead of matrix \( \mathcal{H} \) of dimension \( n \), we investigate in (8) a matrix of dimension \( \nu = (n + 1)n/2 \) corresponding to the operator \( \mathcal{A}_r \). In section 2 we shall show the relation between \( \det \mathcal{H} \) and \( \det \mathcal{A}_r \). In section 3 we generalize the procedure of section 2 to get a solution for polynomial matrices.

2. Relation between Hurwitz- und Lyapunov – Kronecker matrix. We consider the reduced Lyapunov-Kronecker matrix from (7). From (6) we can easily show that the first row of (5) influences only the first \( n \) columns of the matrix \( \mathcal{A}_r \).

\[
\mathcal{A}_r = \begin{bmatrix} A & V \end{bmatrix}
\]

(9)
The matrix $V \in \mathbb{R}^{(n+1)n/2 \times (n-1)n/2}$ results from the subdiagonal of the companion matrix and has full rank. We can find a regular transformation for (9) so that

$$
\begin{bmatrix}
\mathcal{N}' \\
\mathcal{W}'
\end{bmatrix} \cdot A_r = 
\begin{bmatrix}
\tilde{\mathcal{H}} & 0 \\
X & I
\end{bmatrix}
$$

where $\mathcal{N}$ is the kernel of $V'$ and therefore independent of $a_i$ and $\mathcal{W}$ coincides with $V$ except for a normalization. It is clear from (10) that the regularity of $A_r$ is dependent on the regularity of $\mathcal{H}$. To show the dependence explicitly, a possible transformation using elementary matrix operations independent of $a_i$ is given by

$$
E_r \cdot D \cdot A_r \cdot E_r = 
\begin{bmatrix}
\mathcal{H} & 0 \\
X & I
\end{bmatrix}
$$

where $E_r$ includes the elementary row and columns operations, $D$ a diagonal matrix with $\det(D) = 2^{-n}$ and $X$ is a matrix which is not important for our analysis.

Therefore

$$
\det(A_r) = 2^n \det(\mathcal{H})
$$

and the elements of the matrix $P$ are strictly proper rational functions in the parameters $a_i$ whose denominators are the critical stability conditions. The highest element degree in these multivariable polynomials is $\frac{n}{2}$ for even polynomials and $\frac{n-1}{2}$ for odd polynomials. Similar analysis is found in [2] and [3].

3. Generalized Hurwitz Matrix for Polynomial Matrices. In this section we try to get a Hurwitz matrix for polynomial matrices following in analogous way the treatment in section 2.

Consider the polynomial matrix

$$
A(s) = [a_{ij}(s)] \in \mathbb{R}^{m \times m}
$$

whose elements are polynomials in $s$ of maximal degree $n$. Then we have

$$
A(s) = A_0 + A_1 s + A_2 s^2 + \cdots + A_{n-1} s^{n-1} + Is^n
$$

where $A_k \in \mathbb{R}^{m \times m}$, $k = 1, 2, \ldots, n-1$ are real constant matrices.

The stability of $A(s)$ is determined by the application of Hurwitz criterion to the polynomial

$$
d(s) = \det(A(s)).
$$

The Hurwitz matrix thus obtained has elements which are multiaffine functions of the coefficients of the polynomials $a_{ij}(s)$. The dimension of the resulting Hurwitz matrix $\mathcal{H}$ is $\nu = n \cdot m$.  


3.1. Lyapunov Stability. To obtain an affine dependence between the stability criterion description and the coefficients of $A(s)$ we use here the state space realisation in form of the block companion matrix $A_m \in \mathbb{R}^{n \cdot m \times n \cdot m}$

\[
A_m = \begin{bmatrix}
-A_{n-1} & -A_{n-2} & \cdots & -A_1 & -A_0 \\
I & 0 & \cdots & \cdots & \cdots \\
0 & I & 0 & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & I & 0
\end{bmatrix}
\]  

(15)

The block companion matrix $A_m$ has only $m$ distinguished rows. The resulting operator $A_r$ using Kronecker formulation has the dimension $\mu = (n \cdot m + 1) \cdot n \cdot m/2$. For example, for $n = 4$ and $m = 2$ we get $H \in \mathbb{R}^{8 \times 8}$ and $A_r \in \mathbb{R}^{36 \times 36}$ which leads to an eigenvalue problem of considerably high dimension. Now different questions arise: Is it possible to reduce the Lyapunov-Kronecker formulation for $m$-companion matrices? Can we get a Hurwitz matrix for the polynomial matrix where the coefficients of the original polynomials appear explicitly in a simple structure? What are the stability characteristics which can be extracted from this generalized Hurwitz matrix?

For nonmonic (13) we can work with a generalization of the Kronecker formulation using the guardian operator

\[
A' \otimes B' + B' \otimes A'
\]

where

\[
B = \begin{bmatrix}
A_n & 0 & 0 & \cdots & 0 \\
0 & I & 0 & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & I
\end{bmatrix}
\]

3.2. Generalized Hurwitz Matrix for $m=2$. We will explain the procedure of generalization of the Hurwitz matrix for $n = 3$. But for general $n$ the same kind of procedure is valid.

Consider the polynomial matrix

\[
A(s) = \begin{bmatrix}
s^3 + a_5 s^2 + a_3 s + a_1 & a_4 s^2 + a_2 s + a_0 \\
b_5 s^2 + b_3 s + b_1 & s^3 + b_4 s^2 + b_2 s + b_0
\end{bmatrix}
\]

with the characteristic polynomial

\[
a(s) = \det(A(s)) = s^6 + \alpha_5 s^5 + \alpha_4 s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0
\]
where
\[
\begin{align*}
\alpha_5 &= a_5 + b_4 \\
\alpha_4 &= a_3 + b_2 + a_5 b_4 - a_4 b_5 \\
\alpha_3 &= a_1 + b_0 + a_5 b_2 + a_3 b_4 - a_4 b_3 - a_2 b_5 \\
&\vdots \\
\alpha_0 &= a_1 b_0 - a_0 b_1.
\end{align*}
\]

The corresponding Hurwitz matrix $H \in \mathbb{R}^{n \times n}$ is

\[
H_{6} = \begin{bmatrix}
\alpha_{n-1} & \alpha_{n-3} & \alpha_{n-5} & \ldots & \ldots & 0 & 0 \\
1 & \alpha_{n-2} & \alpha_{n-4} & \ldots & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\alpha_{n-1} & \alpha_{n-3} & \alpha_{n-5} & \ldots & \ldots & \alpha_{3} & \alpha_{1} \\
\alpha_{n-2} & \alpha_{n-4} & \ldots & \ldots & \ldots & \alpha_{4} & \alpha_{2} & \alpha_{0}
\end{bmatrix}
\]

and the block companion matrix $A_2$ associated with $A(s)$

\[
A_2 = \begin{bmatrix}
-a_5 & -a_4 & -a_3 & -a_2 & -a_1 & -a_0 \\
-b_5 & -b_4 & -b_3 & -b_2 & -b_1 & -b_0 \\
1 & 0 & \ldots & \ldots & \ldots & \ldots \\
0 & 1 & \ldots & \ldots & \ldots & \ldots \\
\end{bmatrix}
\]

Define

\[
V_1 = \begin{bmatrix}
1 & a_5 & a_3 & a_1 \\
1 & a_5 & a_3 & a_1 \\
\vdots & \vdots & \vdots & \vdots \\
1 & a_5 & a_3 & a_1 \\
\end{bmatrix} \in \mathbb{R}^{6 \times 9}
\]

\[
V_2 = \begin{bmatrix}
a_4 & a_2 & a_0 \\
a_4 & a_2 & a_0 \\
\vdots & \vdots & \vdots \\
a_4 & a_2 & a_0 \\
\end{bmatrix} \in \mathbb{R}^{6 \times 8}
\]
Then

\[
\mathcal{H}_6 = \begin{bmatrix} V_1 & -V_2 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}
\]

and

\[
(16) \quad \det(\mathcal{H}_6) = \det \begin{bmatrix} I & W_1 \\ I & W_2 \\ V_1 & -V_2 \\ 0 & 0 \end{bmatrix}.
\]

It is noted that the coefficients of the polynomials in \( A(s) \) appear explicitly in the matrix (16) whose determinant gives the critical stability condition. Also it is obvious that the leading principal minors of the Hurwitz matrix \( \mathcal{H}_6 \) results also from (16). Therefore (16) gives the complete set of stability conditions. It is easy to show that for any degree \( n \) a similar construction results.

If we use the Lyapunov Kronecker formulation with elementary row operations and scaling we get

\[
E \cdot D \cdot \mathcal{A}_r = \begin{bmatrix} \mathcal{W} & 0 \\ X & I \end{bmatrix}.
\]

This result looks formally similar to the polynomial case. However the dimension of \( \mathcal{W} \in \mathbb{R}^{(4n-1) \times (4n-1)} \) is significantly higher as if we use the characteristic polynomial to compute \( \mathcal{W} \in \mathbb{R}^{2n \times 2n} \). We can construct \( \mathcal{W} \) directly as follows. For the same example \( n = 3 \) define

\[
K = \begin{bmatrix} a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \\ b_5 & b_4 & b_3 & b_2 & b_1 & b_0 \end{bmatrix}'.
\]
and

\[ V = \begin{bmatrix} K & I_2 & 0 \\ 0 & K & I_2 \\ 0 & 0 & K \end{bmatrix} \in \mathbb{R}^{(2n-1) \times m} \]

With this we obtain

\[ \tilde{W} = \begin{bmatrix} D_1 \begin{bmatrix} V \\ 0 \end{bmatrix} ; D_2 \begin{bmatrix} 0 \\ V \end{bmatrix} \end{bmatrix} E \]

where \( D_1, D_2 \) are periodic, diagonal weighting matrices

\[ D_1 = \text{diag}[1, 1, 0, -1, -1, 0, 1, 1, 1, 0] \]

\[ D_2 = \text{diag}[0, 1, 1, 1, 0, -1, -1, -1, 0, 1, 1] \]

and the operator \( E \) cause an addition of the 2\(^{nd} \) and 7\(^{th} \) columns to replace column 2 and eliminate column 7. The \( \tilde{W} \) and \( W \) are identical up to elementary column operations. This gives

\[
\tilde{W} = \begin{bmatrix}
\begin{array}{cccccccc}
  a_5 & b_5 & 1 & 0 & 0 & 0 & 0 & 0 \\
  a_4 & b_4 & 0 & 1 & 0 & 0 & 0 & 0 \\
  a_3 & b_3 & a_5 & b_5 & 1 & 0 & 0 & 0 \\
  a_2 & b_2 & a_4 & b_4 & 0 & 1 & 0 & 0 \\
  a_1 & b_1 & a_3 & b_3 & a_5 & b_5 & 0 & 1 \\
  a_0 & b_0 & a_2 & b_2 & a_4 & b_4 & 0 & 1 \\
  0 & 0 & a_1 & b_1 & a_3 & b_3 & a_5 & b_5 \\
  0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & \cdots & \cdots & a_0 & b_0 \\
\end{array}
\end{bmatrix}
\begin{bmatrix} D_1 \begin{bmatrix} V \\ 0 \end{bmatrix} ; D_2 \begin{bmatrix} 0 \\ V \end{bmatrix} \end{bmatrix} E
\]

\[
\begin{bmatrix}
\begin{array}{cccccccc}
  a_5 & b_5 & 0 & 0 & 0 & 0 & 0 & 0 \\
  a_4 & b_4 & 0 & 1 & 0 & 0 & 0 & 0 \\
  a_3 & b_3 & a_5 & b_5 & 0 & 1 & 0 & 0 \\
  a_2 & b_2 & a_4 & b_4 & 0 & 1 & 0 & 0 \\
  a_1 & b_1 & a_3 & b_3 & a_5 & b_5 & 0 & 1 \\
  a_0 & b_0 & a_2 & b_2 & a_4 & b_4 & 0 & 1 \\
  0 & 0 & a_1 & b_1 & a_3 & b_3 & a_5 & b_5 \\
  0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & \cdots & \cdots & a_0 & b_0 \\
\end{array}
\end{bmatrix}
\]

\[ E \]
This $W$ can be considered as the generalized Hurwitz matrix whose determinant give the critical stability condition. However, it is not yet clear how the complete set of stability conditions can be obtained. For higher values of $n$ we have the same construction. The matrices $D_1$ and $D_2$ are thereby periodic weighting matrices. For $m > 2$ similar construction can be obtained with more complex periodicity.

4. Conclusions. It was shown that a generalized Hurwitz matrix can be obtained for polynomial matrices through construction. The determinant of this matrix gives the critical stability conditions. However how the remaining stability conditions can be retrieved from it is still an open question.

References