A multivariable normal-form for model reduction of discrete-time systems

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The normal form derived by Mansour [1] for single-input / single-output, time-invariant, linear discrete systems will be extended to the multivariable case. An algorithm will be used to achieve simple computation of this form starting with the Luenberger first canonical form. The construction of this form will be illustrated by examples applied to model reduction of multivariable discrete-time systems.

Keywords: Linear multivariable systems, Canonical forms, Discrete-time systems, Model reduction.

1. Introduction

The normal form for single-input / single-output, linear, time-invariant discrete-time systems corresponding to the Schwarz form for continuous systems was first introduced by Mansour [1] to provide a proof of the Schur–Cohn stability criterion for discrete systems through the second method of Lyapunov by choosing a particular Lyapunov function based on that form. The transformations to this form and other normal forms are presented in [11]. This form in a modified version was used by Badreddin and Mansour [2] for model reduction and in its original version by Dourdoumas [3] for identification. Other properties of this form were presented in [4], [5], [10].

The main objective of this paper is to show how this form can be extended to the multivariable case. The resulting multivariable form is mainly developed for model reduction of multivariable discrete-time systems using the quasi-steady state concept shown in [2]. In contrast to the single-input case, where a controllable linear system can uniquely be described by the companion matrix or one of its variants, e.g. the controllability canonical form, the analogous form for the multi-input case depends on the choice of the n-linearly independent vectors from the controllability matrix, where n is the order of the system considered. Efforts were made to find a 'nice' choice which follows according to a certain scheme and to use these vectors in the similarity transformation matrix, which transforms the original system to a particularly nice form. Luenberger [6], Popov [7], Yokoyama and Kinnen [8], Denham [9] and many others proposed 'nice' schemes to choose the transformation base.

The Luenberger first form will be taken here as a starting point for the construction of a new multivariable discrete form, because although non-unique it decomposes the system into a number of subsystems each described by a companion matrix, and the coupling between the subsystems is only in one direction.

The single-input form due to Mansour will first be reviewed. The construction of the multivariable form will be illustrated by examples rather than by detailed proofs.
2. The single-input normal form due to Mansour

The form used for model reduction [2] will be shown again here. Consider the controllable pair $A, b$. We can assume, without loss of generality, that $A, b$ are in the controllability form. For the $n$-th order system,

$$A_0 = \begin{bmatrix}
0 & -a_n \\
1 & -a_{n-1} \\
0 & \ddots & -a_2 \\
\vdots & \ddots & \ddots \\
0 & \ddots & 1 & -a_1
\end{bmatrix}, \quad b = \begin{bmatrix} 1 \\
0 \\
\vdots \\
0
\end{bmatrix}, \quad (1)$$

the single-input normal form will be

$$F_0 = T_0 A_0 T_0^{-1} = \begin{bmatrix}
-\Delta_1 (1 - \Delta_1^2) & \ldots & 0 \\
-\Delta_2 & -\Delta_1 \Delta_2 & \ddots \\
\vdots & \ddots & \ddots \\
-\Delta_n & -\Delta_1 \Delta_n & \ldots & -\Delta_{n-1} \Delta_n
\end{bmatrix}, \quad g = T_0 b = \begin{bmatrix} 1 \\
0 \\
\vdots \\
0
\end{bmatrix}, \quad (2)$$

Since the controllability matrix of (1) is the unity matrix, $[b, A_0 b, A_0^2 b, \ldots, A_0^{n-1} b] = I$, then $T_0$ is the controllability matrix of (2):

$$T_0 = [g, F_0 g, F_0^2 g, \ldots, F_0^{n-1} g]. \quad (3)$$

Note that one could make another choice of $g$ which might be useful in other applications, e.g. for dead-beat controller design, taking $g = [\Delta_1, \Delta_2, \ldots, \Delta_n]$ results in a simple formulation of the feedback gains.

The elements $\Delta_1, \Delta_2, \ldots, \Delta_n$ are obtainable from the scalar Schur–Cohn table which is shown here for a third-degree polynomial for clarity.

Consider the polynomial

$$P(z) = z^3 + a_1 z^2 + a_2 z + a_3;$$

the Schur–Cohn table is constructed as in Table 1.

<table>
<thead>
<tr>
<th>$a_3$</th>
<th>$a_2$</th>
<th>$a_1$</th>
<th>1</th>
<th>$\Delta_1 = a_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1_a_3$</td>
<td>$a_2$</td>
<td>$a_1$</td>
<td>1</td>
<td>$= a_3,3$</td>
</tr>
</tbody>
</table>

$$(a_2 - a_1 a_3) \quad (a_1 - a_2 a_3) \quad (1 - a_3^2) \quad \Delta_2 = \frac{a_2 - a_1 a_3}{(1 - a_3^2)}$$

normalized:

<table>
<thead>
<tr>
<th>$a_2,2$</th>
<th>$a_1,2$</th>
<th>1</th>
<th>$\Delta_1 = a_{1,2} - a_1 a_2 a_3 \quad (1 - a_2,2^2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$a_1,2$</td>
<td>$a_2,2$</td>
<td>$= a_{2,2}$</td>
</tr>
<tr>
<td>$(a_1,2 - a_1 a_2 a_3) \quad (1 - a_2,2^2)$</td>
<td>$\Delta_1 = a_{1,2} - a_1 a_2 a_3 \quad (1 - a_2,2^2)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_{1,1}$</td>
<td>1</td>
<td>$= a_{1,1}$</td>
<td></td>
</tr>
</tbody>
</table>
3. The multi-input normal form

3.1. The Luenberger first form [6]

Given a controllable pair \((A, B)\) of dimensions \((n \times n)\) and \((n \times m)\), respectively. Assume \(B\) has the full rank, and

\[ B = [b_1, b_2, \ldots, b_m]. \]

The Luenberger first form can be derived as follows: Select a column of \(B\), say \(b_1\), and evaluate the vector \(Ab_1\); if it is linearly independent of \(b_1\) retain it and proceed to \(A^2b_1, A^3b_1, \ldots\) until dependency of, say, \(A^r b_1\) all the preceding vectors arises. Take another column of \(B\), say, \(b_2\), and repeat the above procedure until \(n\)-linearly independent vectors are chosen in this manner. These vectors will be ordered in the matrix

\[ P = [b_1, Ab_1, \ldots, A^{(r_1-1)}b_1, b_2, Ab_2, \ldots, A^{(r_2-1)}b_2, \ldots, b_m, Ab_m, \ldots, A^{(r_m-1)}b_m], \]

\[ \sum_{i=1}^{m} r_i = n. \]

The similarity transformation,

\[ A_c = P^{-1}AP, \quad B_c = P^{-1}B, \]

where

\[ A_c = [A_{ij}] \quad \text{and} \quad B_c = [b_{1c}, b_{2c}, \ldots, b_{mc}], \]

will result in

\[ A_{ii} = \begin{bmatrix} 0 & \ldots & 0 & \times \\ 1 & 0 & \ldots & 0 & \times \\ 0 & 1 & \ddots & \times \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \ldots & 1 & \times \end{bmatrix}, \quad A_{ij} = \begin{bmatrix} 0 & \ldots & 0 & \times \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \ldots & 0 & \times \end{bmatrix} \quad \text{for } i < j, \quad A_{ij} = 0 \text{ for } i > j. \]

The dimension of \(A_{ii}\) is \((r_i \times r_i)\). \(b_{ic}\) is an \(n\)-dimensional unit vector with 1 in the \(i\)-th position and zeros everywhere else.

For the columns of \(B\) which do not appear in the matrix \(P\), the corresponding columns of \(B_c\) will have no special structure. In the following it will be assumed that all columns of \(B\) will appear in \(P\). The dimension of \(A_{ij}\) is \((r_i \times r_i)\), and \(\times\) denotes a possible non-zero element.

The Luenberger first form is not unique in the sense that the permutation of the columns of \(B\) may result in other dimensions \(r_i\) for the diagonal blocks \(A_{ii}\) of \(A_c\). It possesses, however, the unique property of decomposing the system in a number of subsystems with the coupling between them going in only one direction.

3.2. The proposed first multivariable normal form

In the following a multivariable normal form for discrete-time systems, based on the Luenberger first form, will be proposed to be used for model reduction of discrete-time systems.

By means of elementary similarity transformations, \((A_c, B_c)\) could be brought to the following lower-block-triangular form which we denote by \((A, B)\) for simplicity:
where \( \times \) denotes nonspecific elements and empty spaces represent zero elements. The \( \times \)'s within each diagonal block are the coefficients of the characteristic polynomial of the corresponding subsystem.

In the following, the \( \times \)'s below the diagonal blocks will be denoted by \( \beta_j \) and they represent the coupling between the subsystems.

An obvious extension of the single input normal form to the multivariable case could be \( (F, G) \).

The diagonal blocks are single-input normal form representations of the corresponding companion diagonal blocks in (4). The \( \times \)'s below the diagonal blocks will be denoted by \( \delta_i \). \( (F, G) \) will be similar to \( (A, B) \) iff there exists a nonsingular matrix \( T \) so that

\[
F = TAT^{-1}, \quad G = TB.
\]

Equation (6) will determine how the coupling elements \( \delta_i \) between the subsystems in (5) are related to the \( \beta_j \) in (4). But since the number of \( \delta_i \) is larger than \( \beta_j \), there will be some freedom in choosing \( \delta_i \), i.e. there exist interrelations between the \( \delta_i \)'s. One could make use of this interrelation to achieve a particularly nice structure of \( F \) and/or easy computation of \( \delta_i \) given \( \beta_j \).

Starting by the system description \( (A, B) \) of (4), let the size of the diagonal block \( A_{ii} \) be \( \mu_i \) and \( A_{ij} = [0, \ldots, 0, \beta_j] \) where \( \beta_j \) is a vector of compatible dimension. The size of \( F_{ii} \) in (5) is equal to the size of \( A_{ii} \) and coupling elements will be represented by the vectors \( \delta_i \) and \( G = B \).

The proposed multivariable form will then be as follows:
For 2 inputs \((m = 2)\):

\[
A = \begin{bmatrix}
A_{11} & 0 \\
0 & A_{22}
\end{bmatrix},
\quad F = \begin{bmatrix}
F_{11} \\
\begin{bmatrix}
\delta_1, -\Delta_1 \delta_1, -\Delta_2 \delta_1, \ldots, -\Delta(\mu_1-1) \delta_1
\end{bmatrix}
\end{bmatrix},
\quad 0
\]

\[
T = \begin{bmatrix}
g_2, \ldots, g_{\mu_1}^{-1} g_2, g_1, F g_1, \ldots, F_{\mu_2}^{-1} g_1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
T_{11} & 0 \\
T_{21} & T_{22}
\end{bmatrix}
\]

where \(g_i\) is the \(i\)-th column of \(G\),

\[
\delta_1 = F_{22}^{-1(\mu_1-1)} T_{22} \beta_1.
\] (7)

For 3 inputs \((m = 3)\):

\[
A = \begin{bmatrix}
A_{11} & A_{22} \\
\begin{bmatrix} 0 \ldots 0 \beta_1 \end{bmatrix} & A_{33}
\end{bmatrix},
\quad F = \begin{bmatrix}
F_{11} \\
\begin{bmatrix}
-\delta_1, -\Delta_1 \delta_1, -\Delta_2 \delta_1, \ldots, -\Delta(\mu_1-1) \delta_1
\end{bmatrix}
\end{bmatrix},
\quad 0
\]

\[
T = \begin{bmatrix}
g_3, \ldots, g_{\mu_1}^{-1} g_3, g_2, F g_2, \ldots, F_{\mu_2}^{-1} g_2, g_1, \ldots, F_{\mu_3}^{-1} g_1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
T_{11} & 0 \\
T_{21} & T_{22} & 0 \\
T_{31} & T_{32} & T_{33}
\end{bmatrix}
\]

\[
\delta_2 = F_{33}^{-1(\mu_2-1)} T_{33} \cdot \beta_2,
\] (8.1)

\[
\delta_1 = \begin{bmatrix}
F_{22} \\
\begin{bmatrix}
-\delta_2, -\Delta_2 \delta_2, \ldots
\end{bmatrix}
\end{bmatrix}^{-1(\mu_1-1)} \cdot \begin{bmatrix}
T_{22} & 0 \\
T_{32} & T_{33}
\end{bmatrix} \cdot \beta_1.
\] (8.2)

One starts by evaluating the coupling elements for the lowest diagonal block and then proceeds upwards, since the coupling elements for a diagonal block will depend on those of lower blocks as seen from the above example.

To extension to an arbitrary number of inputs is easily attained according to the above construction scheme.

The proof for the similarity of the pairs \((A, B)\) and \((F, G)\) with the transformation matrix \(T\) is given in the Appendix.

However, the following example should illustrate the main idea of that, quite lengthy, proof as well as the application to model reduction using the method of [2].
Example 1. Suppose

\[
A = \begin{bmatrix}
0 & -a_{21} & 0 & 0 & 0 \\
1 & -a_{11} & 0 & 0 & 0 \\
0 & -\beta_3 & 0 & 0 & -a_{32} \\
0 & -\beta_2 & 1 & 0 & -a_{22} \\
0 & -\beta_1 & 0 & 1 & -a_{12}
\end{bmatrix} = \begin{bmatrix}
A_{11} & 0 \\
A_{21} & A_{22}
\end{bmatrix},
\]

\[
F = \begin{bmatrix}
\Delta_{21} & (1 - \Delta_{11}^2) & 0 & 0 & 0 \\
\Delta_{21} & -\Delta_{11} \Delta_{21} & 0 & 0 & 0 \\
-\delta_3 & -\delta_6 & -\Delta_{12} & (1 - \Delta_{12}^2) & 0 \\
-\delta_2 & -\delta_5 & -\Delta_{22} & -\Delta_{12} \Delta_{22} & (1 - \Delta_{22}^2) \\
-\delta_1 & -\delta_4 & -\Delta_{32} & -\Delta_{12} \Delta_{32} & -\Delta_{22} \Delta_{32}
\end{bmatrix},
\]

\[
G = \begin{bmatrix}
0 & 1 \\
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} = \begin{bmatrix} b_1, b_2 \end{bmatrix},
\]

Since \([b_2, Ab_2, b_1, Ab_1, A^2b_1] = I\) and \(I\) is the \(5 \times 5\) unity matrix let

\[
T = \begin{bmatrix}
g_2, Fg_2, g_1, Fg_1, F^2g_1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & -\Delta_{11} & 0 & 0 & 0 \\
0 & -\Delta_{21} & 0 & 0 & 0 \\
0 & -\delta_1 & 1 & -\Delta_{12} & t_1 \\
0 & -\delta_2 & 0 & -\Delta_{22} & t_2 \\
0 & -\delta_3 & 0 & -\Delta_{32} & t_3
\end{bmatrix},
\]

where

\[
t_1 = \Delta_{12} a_{1,2} - \Delta_{22}, \quad a_{1,2} = \Delta_{12} + \Delta_{12} \Delta_{22},
\]

\[
t_2 = \Delta_{22} a_{1,2} - \Delta_{32}, \quad t_3 = \Delta_{32} a_{1,2},
\]

\[
FT = TA,
\]

\[
\frac{F_{11}T_{11}}{(F_{11}T_{11} + F_{22}T_{21})} = \begin{bmatrix} 0 \\
T_{11} & 0 \\
T_{21} & T_{22}
\end{bmatrix},
\]

From the direct analogy of the single input case, it is evident that \(F_{11}T_{11} = T_{11}A_{11}\) and \(F_{22}T_{22} = T_{22}A_{22}\) since \(T_{11}, T_{22}\) are the controllability matrices of the 1st and 2nd subsystems, respectively.

Now the equation

\[
F_{21}T_{11} + F_{22}T_{21} = T_{21}A_{11} + T_{22}A_{21}
\]  \hspace{1cm} (9)

must be satisfied. Notice that \(F_{11}, T_{11}, F_{22}, T_{22}\) do not contain any of the coupling elements \(\delta_i\) or \(\beta_j\), and (9) alone determines the relation between \(\delta_i\) and \(\beta_j\).
It is easy to see that there are more $\delta_i$ in (9) than the number of equations available. This suggests that some $\delta_i$ could be chosen to fulfill certain wishes.

Let $\delta_6 = \Delta_{11}\delta_3$, $\delta_5 = \Delta_{11}\delta_2$, $\delta_4 = \Delta_{11}\delta_1$. The solution of (9) will give

$$
\begin{bmatrix}
\delta_3 \\
\delta_2 \\
\delta_1 
\end{bmatrix} = F_{22}^{-1}T_{22}
\begin{bmatrix}
\beta_3 \\
\beta_2 \\
\beta_1 
\end{bmatrix}.
$$

(10)

The above choice has the following advantages:

1. The structure of $F_{21}$ matches that of the last row of $F_{11}$.
2. The computation of $\delta_i$ is straight-forward, since $F_{22}$ has the normal form (2) whose inverse can be evaluated directly by the expressions (A.21) or (A.22) in the Appendix.
3. The structure of $F_{21}$ will not be changed when the model reduction algorithm described in [2] is used.

This last property is further explained in the following.

3.3. Reduction of the first normal form

Example 2. Consider the pair $(F, G)$ of Example 1:

$$
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 
\end{bmatrix}_{k+1} =
\begin{bmatrix}
-\Delta_{11} & (1 - \Delta_{12}^2) & 0 & 0 & 0 \\
-\Delta_{21} & -\Delta_{11}\Delta_{21} & 0 & 0 & 0 \\
-\delta_6 & -\Delta_{11}\delta_1 & -\Delta_{12} & (1 - \Delta_{12}^2) & 0 \\
-\delta_5 & -\Delta_{11}\delta_2 & -\Delta_{22} & -\Delta_{12}\Delta_{22} & (1 - \Delta_{22}^2) \\
-\delta_4 & -\Delta_{11}\delta_1 & -\Delta_{32} & -\Delta_{12}\Delta_{32} & -\Delta_{22}\Delta_{32}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 
\end{bmatrix}_k +
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 
\end{bmatrix}_k
$$

and the outputs

$$
\begin{bmatrix}
y_1 \\
y_2 
\end{bmatrix}_k =
\begin{bmatrix}
h_{11} & h_{12} & h_{13} & h_{14} & h_{15} \\
h_{21} & h_{22} & h_{23} & h_{24} & h_{25}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 
\end{bmatrix}_k.
$$

Case 1: Reduction of the 2nd subsystem (lower-block).

$$
\dot{x}_3(k) = -\frac{\delta_3x_1(k)}{1 + \Delta_{22}\Delta_{32}} - \Delta_{11}\frac{\delta_3x_2(k)}{1 + \Delta_{22}\Delta_{32}} - \Delta_{32}x_3(k) - \Delta_{12}\Delta_{32}x_4(k).
$$

Substituting for $x_5$ by $\hat{x}_5$ in the dynamic equations of $x_3$ and $x_4$ ($x_1$, $x_2$ are independent of $x_5$), we obtain

$$
\dot{x}_3(k + 1) = -\delta_6x_1(k) - \Delta_{11}\delta_2x_2(k) - \Delta_{12}\dot{x}_3(k) - (1 - \Delta_{12}^2)\dot{x}_4(k) + u_1(k),
$$
$$
\dot{x}_4(k + 1) = -\delta_5x_1(k) - \Delta_{11}\delta_1x_2(k) - \Delta_{22}\dot{x}_3(k) - \Delta_{12}\Delta_{22}\dot{x}_4(k),
$$

where

$$
\delta_2 = \delta_2 + \frac{\delta_3}{1 + \Delta_{22}\Delta_{32}} = \beta_3 - \hat{\Delta}_{22}\beta_1, \quad \hat{\Delta}_{22} = \frac{\Delta_{22} + \Delta_{32}}{1 + \Delta_{22}\Delta_{32}}.
$$
The reduced model may then be written as

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \dot{x}_3 \\
  \dot{x}_4
\end{bmatrix}_{k+1} =
\begin{bmatrix}
  -\Delta_{11} & (1 - \Delta_{11}^2) & 0 & 0 \\
  -\Delta_{21} & -\Delta_{11} \Delta_{21} & 0 & 0 \\
  -\delta_1 & -\Delta_{11} \delta_1 & -\Delta_{12} & (1 - \Delta_{12}^2) \\
  -\delta_2 & -\Delta_{11} \delta_2 & -\Delta_{22} & -\Delta_{12} \Delta_{22}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \dot{x}_3 \\
  \dot{x}_4
\end{bmatrix}_k
+ \begin{bmatrix}
  0 & 1 \\
  0 & 0 \\
  0 & 0 \\
  0 & 0
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix}_k.
\]

The output equation will be

\[
\begin{bmatrix}
  \dot{y}_1 \\
  \dot{y}_2
\end{bmatrix}_k =
\begin{bmatrix}
  (h_{11} - \tilde{h}_1) & (h_{12} - \Delta_{11} \tilde{h}_1) & (h_{13} - \tilde{h}_1) & (h_{14} - \Delta_{12} \tilde{h}_1) \\
  (h_{21} - \tilde{h}_2) & (h_{22} - \Delta_{11} \tilde{h}_2) & (h_{13} - \tilde{h}_2) & (h_{24} - \Delta_{12} \tilde{h}_2)
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \dot{x}_3 \\
  \dot{x}_4
\end{bmatrix}_k
\]

where

\[
\tilde{h}_1 = \frac{\delta_3}{1 + \Delta_{22} \Delta_{32}} h_{15}, \quad h_1 = \frac{\Delta_{32}}{1 + \Delta_{22} \Delta_{32}} h_{15}, \quad \tilde{h}_2 = \frac{\delta_3}{1 + \Delta_{22} \Delta_{32}} h_{25}, \quad h_2 = \frac{\Delta_{32}}{1 + \Delta_{22} \Delta_{32}} h_{25}.
\]

**Remark.** Since

\[
\begin{bmatrix}
  \delta_1 \\
  \delta_2 \\
  \delta_3
\end{bmatrix}_k =
\begin{bmatrix}
  F_{22}^{-1} T_{22} & \beta_1 \\
  \beta_2 \\
  \beta_3
\end{bmatrix} =
\begin{bmatrix}
  F_{22}^{-1} g_{22}, F_{22} g_{22}, F_{22}^2 g_{22}
\end{bmatrix}
\begin{bmatrix}
  \beta_1 \\
  \beta_2 \\
  \beta_3
\end{bmatrix},
\]

then

\[
\begin{bmatrix}
  \delta_1 \\
  \delta_2 \\
  \delta_3
\end{bmatrix} =
\begin{bmatrix}
  F_{22}^{-1} g_{22}, F_{22} g_{22}, F_{22}^2 g_{22}
\end{bmatrix}
\begin{bmatrix}
  \beta_1 \\
  \beta_2 \\
  \beta_3
\end{bmatrix} =
\begin{bmatrix}
  -\Delta_{12} & 1 & -\Delta_{12} \\
  1 & 0 & -\Delta_{22} \\
  0 & 0 & -\Delta_{22}
\end{bmatrix}
\begin{bmatrix}
  \beta_1 \\
  \beta_2 \\
  \beta_3
\end{bmatrix}.
\]

The coupling elements of the reduced model are computed as

\[
\begin{bmatrix}
  \delta_1 \\
  \delta_2
\end{bmatrix} =
\begin{bmatrix}
  F_{22}^{-1} g_{22}, F_{22} g_{22}, F_{22}^2 g_{22}
\end{bmatrix}
\begin{bmatrix}
  \beta_1 \\
  \beta_2 \\
  \beta_3
\end{bmatrix} =
\begin{bmatrix}
  -\Delta_{12} & 1 & -\Delta_{12} \\
  1 & 0 & -\Delta_{22} \\
  0 & 0 & -\Delta_{22}
\end{bmatrix}
\begin{bmatrix}
  \beta_1 \\
  \beta_2 \\
  \beta_3
\end{bmatrix},
\]

i.e. \( \delta_1 = \delta_1, \delta_2 = \beta_3 - \Delta_{22} \beta_1 \).

**Case 2:** Reduction of the 1st subsystem (upper block).

\[
\dot{x}_2(k) = \frac{-\Delta_{21}}{1 + \Delta_{11} \Delta_{21}} \dot{x}_1(k),
\]

and by substitution in the rest of dynamic equations we obtain the reduced model

\[
\begin{bmatrix}
  \dot{x}_1 \\
  \dot{x}_3 \\
  \dot{x}_4 \\
  \dot{x}_5
\end{bmatrix}_{k+1} =
\begin{bmatrix}
  -\tilde{\Delta}_{11} & 0 & 0 & 0 \\
  -\tilde{\delta}_1 & -\Delta_{12} & (1 - \Delta_{12}^2) & 0 \\
  -\tilde{\delta}_2 & -\Delta_{12} \Delta_{22} & (1 - \Delta_{12}^2) & 0 \\
  -\tilde{\delta}_3 & -\Delta_{12} \Delta_{32} & -\Delta_{12} \Delta_{33} & 0
\end{bmatrix}
\begin{bmatrix}
  \dot{x}_1 \\
  \dot{x}_3 \\
  \dot{x}_4 \\
  \dot{x}_5
\end{bmatrix}_k
+ \begin{bmatrix}
  0 & 1 \\
  0 & 0 \\
  0 & 0 \\
  0 & 0
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix}_k.
\]

where

\[
\tilde{\Delta}_{11} = \frac{\Delta_{11} + \Delta_{21}}{1 + \Delta_{11} \Delta_{21}}, \quad \tilde{\delta}_1 = \frac{\delta_1}{1 + \Delta_{11} \Delta_{21}}, \quad \tilde{\delta}_2 = \frac{\delta_2}{1 + \Delta_{11} \Delta_{21}}, \quad \tilde{\delta}_3 = \frac{\delta_3}{1 + \Delta_{11} \Delta_{21}}.
\]
The output equation will be
\[
\begin{align*}
\begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{bmatrix}
&= 
\begin{bmatrix}
(h_{11} - \tilde{h}_1) & h_{13} & h_{14} & h_{15} \\
(h_{21} - \tilde{h}_2) & h_{23} & h_{24} & h_{25}
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_3 \\
\dot{x}_4 \\
\dot{x}_5
\end{bmatrix}
\end{align*}
\]
where \( \tilde{h}_1 = \frac{\Delta_{21}}{1 + \Delta_{11} \Delta_{21}} h_{21} \), \( \tilde{h}_2 = \frac{\Delta_{21}}{1 + \Delta_{11} \Delta_{21}} h_{22} \).

4. Conclusions and final remarks

An extension of the single-input form due to Mansour [1] to the multivariable case has been presented. The proposed form (termed the 'first-multivariable form') possesses particularly nice structural and computational properties for model reduction of discrete-time systems using the method of [2]. An example for using this form in model reduction has also been included.

The output matrix was not considered in the previous development since it has no influence on the algorithm and no special structure. The transformation of the output matrix is given once the similarity transformation matrix is obtained. Beside model reduction for which this form is specially developed, it might also be useful in other applications analogous to the single-input case [3], [4], [5], [10].

Appendix

The proof for the similarity of the pairs \((A, B)\) and \((F, G)\) with the transformation matrix \(T\) may be established by construction and induction as follows. Consider the 3-input case,
\[
A = \begin{bmatrix}
A_{11} & 0 & 0 \\
A_{21} & A_{22} & 0 \\
A_{31} & A_{32} & A_{33}
\end{bmatrix},
B = \begin{bmatrix}
0 & 0 & b_{33} \\
0 & b_{22} & 0 \\
b_{11} & 0 & 0
\end{bmatrix},
\]
\[
F = \begin{bmatrix}
F_{11} & 0 & 0 \\
F_{21} & F_{22} & 0 \\
F_{31} & F_{32} & F_{33}
\end{bmatrix},
G = \begin{bmatrix}
0 & 0 & g_{33} \\
0 & g_{22} & 0 \\
g_{11} & 0 & 0
\end{bmatrix},
\]
\[
T = \begin{bmatrix}
T_{11} & 0 & 0 \\
T_{21} & T_{22} & 0 \\
T_{31} & T_{32} & T_{33}
\end{bmatrix},
\]
where
\[
\text{dim}[A_{11}] = \text{dim}[F_{11}] = \text{dim}[T_{11}] = \mu_1,
\]
\[
\text{dim}[A_{22}] = \text{dim}[F_{22}] = \text{dim}[T_{22}] = \mu_2,
\]
\[
\text{dim}[A_{33}] = \text{dim}[F_{33}] = \text{dim}[T_{33}] = \mu_3,
\]
and \(\text{dim}[\cdot]\) denotes the dimension of the matrix between brackets, e.g. \(A_{11}\) is a \(\mu_1 \times \mu_1\) matrix.
The vector \(b_{ii}\) and \(g_{ii}\) are unit vectors with compatible dimensions. Now we ought to prove that
\[
FT = TA,
\]
i.e.
\[
\begin{bmatrix}
F_{11}T_{11} & 0 & 0 \\
(F_2T_{11} + F_2T_{11}) & F_{22} & 0 \\
(F_3T_{21} + F_3T_{21} + F_3T_{31}) & (F_3T_{22} + F_3T_{32}) & F_{33}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
T_{11}A_{11} & 0 & 0 \\
(T_2A_{11} + T_2A_{21}) & T_{22}A_{22} & 0 \\
(T_3A_{21} + T_3A_{21} + T_3A_{31}) & (T_3T_{22} + T_3T_{32}) & T_{33}A_{33}
\end{bmatrix}
\]

(A.4)

Since \(T_{ii} = \left[ g_{ii}, F_{ii}g_{ii}, F_{ii}^2g_{ii}, \ldots, F_{ii}^{(\mu_i-1)}g_{ii} \right] \), we know from the single-input case that \(F_{ii}T_{ii} = T_{ii}A_{ii} \) for \( i = 1, 2, 3 \).

Next, we are going to prove that

\[
F_{32}T_{22} + F_{33}T_{32} = T_{32}A_{22} + T_{33}A_{32}.
\]

(A.5)

The transformation matrix \( T \) may be written as

\[
T = \left[ g_3, Fg_3, F^2g_3, \ldots, F^{(\mu_3-1)}g_3, g_2, Fg_2, F^2g_2, \ldots, F^{(\mu_2-1)}g_2, g_1, Fg_1, F^2g_1, \ldots, F^{(\mu_1-1)}g_1 \right]
\]

where \( g_i \) is the \( i \)-th column of the \( G \) matrix in (A.2). Consequently, we may write

\[
\begin{bmatrix}
0 \\
\end{bmatrix}
[
\begin{bmatrix}
0 & 0 & 0 & \cdots \\
0 & g_{22} & F_{22}g_{22} & F_{22}^2g_{22} \\
0 & F_{32}g_{22} & (F_{32}F_{22}g_{22} + F_{33}F_{32}g_{22}) & \cdots & (F_{32}F_{32}g_{22} + F_{33}(F_{32}F_{22} + F_{33}F_{33})g_{22})
\end{bmatrix}
\]

(A.6)

By inspection of the structure of (A.6) we could deduce a general expression for \( T_{32} \) as follows:

\[
T_{32} = \begin{bmatrix}
F_{32}g_{22}, F_{32}F_{22}g_{22}, \ldots, F_{32}F_{(\mu_2-1)}g_{22} \\
0, F_{33}F_{22}g_{22}, F_{33}F_{32}g_{22}, \ldots, F_{33}(F_{32}F_{22} + F_{33}F_{33})g_{22}
\end{bmatrix}
\]

which may be rewritten as

\[
T_{32} = F_{32} \left[ 0, g_{22}, F_{22}g_{22}, \ldots, F_{22}^{(\mu_2-1)}g_{22} \right] \\
+ F_{33} \left[ 0, 0, g_{22}, F_{22}g_{22}, \ldots, F_{22}^{(\mu_2-3)}g_{22} \right] \\
+ F_{33}^2 \left[ 0, 0, 0, g_{22}, F_{22}g_{22}, \ldots, F_{22}^{(\mu_2-4)}g_{22} \right] \\
+ \cdots + F_{33}^{(\mu_2-2)} \left[ 0, \ldots, 0, g_{22}, F_{22}g_{22}, \ldots, F_{22}^{(\mu_2-2)}g_{22} \right]
\]

(A.7)

In order to simplify the above expression we introduce the 'shifting' matrix \( S \) of dimension \( \mu_2 \):

\[
S = \begin{bmatrix}
0 \\
\vdots \\
I_{(\mu_2-1)} \\
0 \\
0 & \ldots & 0
\end{bmatrix}
\]

(A.8)
where $I_k$ denotes the $k$-th dimensional unity matrix. Accordingly,

$$
S^2 = \begin{bmatrix}
0 & 0 & I_{(\mu_2-2)} \\
\vdots & \vdots & \vdots \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad S^j = \begin{bmatrix}
0 & \ldots & 0 & I_{(\mu_2-j)} \\
\vdots & \vdots & \vdots & \vdots \\
0 & \ldots & 0 & 0 \\
0 & \ldots & 0 & 0
\end{bmatrix},
$$

$$
S^{(\mu_2-1)} = \begin{bmatrix}
0 & \ldots & 0 & 1 \\
0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & \ldots & 0 & 0
\end{bmatrix}.
$$

Using the $S$ matrix, we may rewrite (A.7) as

$$
T_{32} = F_{32} T_{22} S + F_{33} F_{32} T_{22} S^2 + F_{33}^2 F_{32} T_{22} S^3 + \ldots + F_{33}^{(\mu_2-2)} F_{32} T_{22} S^{(\mu_2-1)}. \tag{A.9}
$$

We also know that $A_{22}$ is structured as follows:

$$
A_{22} = \begin{bmatrix}
0 & \ldots & 0 & -a_{\mu_2} \\
\vdots & \vdots & \vdots & \vdots \\
I_{\mu_2-1} & & -a_1
\end{bmatrix}
$$

where $a_i$ are the coefficients of the characteristic polynomial of $A_{22}$. Denote the vector $[-a_1, -a_\mu_2, \ldots, -a_1]^T$ by $\rho_i$, then $A_{22}$ may be rewritten as

$$
A_{22} = \begin{bmatrix}
0 & \ldots & 0 & -a_{\mu_2} \\
\vdots & \vdots & \vdots & \vdots \\
I_{\mu_2-1} & & -a_1
\end{bmatrix} \tag{A.10}
$$

Premultiplying $A_{22}$ by $S$ results in

$$
SA_{22} = \begin{bmatrix}
I_{\mu_2-1} & \rho_{\mu_2-1} \\
0 & \ldots & 0 & 0
\end{bmatrix}, \quad S^2 A_{22} = \begin{bmatrix}
0 & I_{\mu_2-2} & \rho_{\mu_2-2} \\
\vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0
\end{bmatrix}, \tag{A.11}
$$

$$
S'A_{22} = \begin{bmatrix}
0 & \ldots & 0 & I_{\mu_2-j} & \rho_{\mu_2-j} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \ldots & 0 & 0 & 0 \\
0 & \ldots & 0 & 0 & 0
\end{bmatrix}.
$$
In particular

\[ S^{(\mu_2-1)}A_{22} = \begin{bmatrix} 0 & \ldots & 0 & 1 & -a_1 \\ 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 & 0 \end{bmatrix} \]

Notice also that postmultiplying \( T_{22} \) by \( S \) results in shifting the columns of \( T_{22} \) by one position to the right and in a zero first column of the resulting matrix.

We denote the first \( k \) columns of \( T_{22} \) by \( T_k \) and \( i \)-th column of \( T_{22} \) by \( c_i \). Notice that \( T_1 = c_1 \).

Equipped by the structural properties obtained above by (A.8) through (A.11) and the definitions of \( T_k \) and \( c_i \), we may rewrite (A.5) as follows:

\[
F_{32} \begin{bmatrix} \Gamma_{\mu_2-1}, c_{\mu_2} \end{bmatrix} + F_{33} F_{32} \begin{bmatrix} 0, \Gamma_{\mu_2-1} \end{bmatrix} + F_{33}^2 F_{32} \begin{bmatrix} 0, 0, \Gamma_{\mu_2-2} \end{bmatrix} + \cdots + F_{33}^{(\mu_2-1)} F_{32} \begin{bmatrix} 0, \ldots, 0, \Gamma_1 \end{bmatrix} \\
= F_{32} \begin{bmatrix} \Gamma_{\mu_2-1}, T_{22} p_{\mu_2-1} \end{bmatrix} + F_{33} F_{32} \begin{bmatrix} 0, \Gamma_{\mu_2-2}, T_{22} p_{\mu_2-2} \end{bmatrix} + \cdots \\
+ F_{33}^j F_{32} \begin{bmatrix} 0, \ldots, 0, \Gamma_{\mu_2-j-1}, T_{22} p_{\mu_2-j-1} \end{bmatrix} + \cdots \\
+ F_{33}^{(\mu_2-2)} F_{32} \begin{bmatrix} 0, \ldots, 0, \Gamma_1, T_{22} p_1 \end{bmatrix} + T_{33} A_{32}.
\]

Eliminating the common columns in both sides of the above equation, we obtain

\[
F_{32} \begin{bmatrix} 0, \ldots, 0, c_{\mu_2} \end{bmatrix} + F_{33} F_{32} \begin{bmatrix} 0, \ldots, 0, c_{\mu_2-1} \end{bmatrix} + F_{33}^2 F_{32} \begin{bmatrix} 0, \ldots, 0, c_{\mu_2-2} \end{bmatrix} + \cdots \\
+ F_{33}^{(\mu_2-2)} F_{32} \begin{bmatrix} 0, \ldots, 0, c_2 \end{bmatrix} + F_{33}^{(\mu_2-1)} F_{32} \begin{bmatrix} 0, \ldots, 0, c_1 \end{bmatrix} \\
= F_{32} \begin{bmatrix} 0, \ldots, 0, T_{22} p_{\mu_2-1} \end{bmatrix} + F_{33} F_{32} \begin{bmatrix} 0, \ldots, 0, T_{22} p_{\mu_2-2} \end{bmatrix} + \cdots \\
+ F_{33} F_{32} \begin{bmatrix} 0, \ldots, 0, T_{22} p_{\mu_2-j-1} \end{bmatrix} + F_{33}^{(\mu_2-2)} F_{32} \begin{bmatrix} 0, \ldots, 0, T_{22} p_1 \end{bmatrix} + T_{33} \begin{bmatrix} 0, \ldots, 0, \beta_2 \end{bmatrix}.
\]

Since \( A_{32} = [0, \ldots, 0, \beta_2] \), this reduces further to

\[
F_{32} F_{22}^{(\mu_2-1)} g_{22} + F_{33} F_{22} F_{22}^{(\mu_2-2)} g_{22} + \cdots + F_{33}^{(\mu_2-1)} F_{22} g_{22} + F_{33}^{(\mu_2-1)} F_{32} g_{22} \\
= F_{33} \Gamma_2 p_{\mu_2-1} + F_{33} F_{32} \Gamma_2 p_{\mu_2-2} + \cdots + F_{33}^{(\mu_2-2)} F_{32} \Gamma_1 p_1 + T_{33} \beta_2.
\]

(A.12)

Up to now we did not impose any restriction on the structure of \( F_{32} \). Taking

\[
F_{32} = \delta_2 \begin{bmatrix} -1, -\Delta_1, -\Delta_2, \ldots, -\Delta_{\mu_2-1} \end{bmatrix}
\]

as proposed, results in

\[
F_{32} F_{22}^i g_{22} = F_{33} \Gamma_i p_i, \quad i = 1, 2, 3, \ldots, (\mu_2 - 1).
\]

(A.14)

Substituting from (A.14) into (A.12) we get \( F_{33}^{(\mu_2-1)} \delta_2 = T_{33} \beta_2 \), or

\[
\delta_2 = F_{33}^{(\mu_2-1)} T_{33} \beta_2
\]

(A.15)

which is the proposed formula for evaluating \( \delta_2 \).

The reason why (A.14) holds, is the special structure of \( F_{22} \) and the relation between the coefficients of the characteristic polynomial \( a_i \) and the Schur–Cohn coefficients \( \Delta_i \). The general expression is given in [1].
but we may illustrate this property by considering the example for which $\mu_2 = 3$.

\[
F_{22} = \begin{bmatrix}
-\Delta_1 & (1 - \Delta_1^2) & 0 \\
-\Delta_2 & -\Delta_1\Delta_2 & (1 - \Delta_2^2) \\
-\Delta_3 & -\Delta_1\Delta_3 & -\Delta_2\Delta_3
\end{bmatrix},
g_{22} = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}.
\]

For $i = 1$,

\[
F_{22}g_{22} = [-\Delta_1, -\Delta_2, -\Delta_3]^T.
\]

The left-hand side (LHS) of (A.14) is

\[
F_{32}F_{22}g_{22} = \delta_2 \begin{bmatrix}
-1, -\Delta_1, -\Delta_2 \\
-\Delta_2 \\
-\Delta_3
\end{bmatrix} = \delta_2 (\Delta_1 + \Delta_1\Delta_2 + \Delta_2\Delta_3),
\]

\[
F_{32}F_{22}g_{22} = \delta_2 a_1, \quad \Gamma_1 = [g_{22}] = [1, 0, 0]^T, \quad p_1 = -a_1.
\]  

The right-hand side (RHS) of (A.14) will be

\[
F_{32}\Gamma_1 p_1 = \delta_2 \begin{bmatrix}
1 \\
0
\end{bmatrix} (-a_1) = \delta_2 a_1,
\]

\[
(A.16.1)
\]

i.e. LHS = RHS.

For $i = 2$,

\[
F_{22}^2g_{22} = \begin{bmatrix}
\Delta_1 a_1 - \Delta_1\Delta_2\Delta_3 \\
\Delta_2 a_1 - \Delta_3 \\
\Delta_3 a_1
\end{bmatrix},
\]

\[
LHS = \delta_2 \begin{bmatrix}
-1, -\Delta_1, -\Delta_2 \\
-\Delta_2 \\
-\Delta_3
\end{bmatrix} F_{22}^2g_{22} = \delta_2 (a_1^2 + a_2),
\]  

\[
(A.17.1)
\]

\[
\Gamma_2 = [g_{22}, F_{22}g_{22}] = \begin{bmatrix}
1 & -\Delta_1 \\
0 & -\Delta_2 \\
0 & -\Delta_3
\end{bmatrix}, \quad p_2 = \begin{bmatrix}
a_2 \\
-a_1
\end{bmatrix}, \quad \Gamma_2 p_2 = \begin{bmatrix}
\Delta_1 a_1 - a_2 \\
\Delta_2 a_1 \\
\Delta_3 a_1
\end{bmatrix},
\]

\[
RHS = F_{32}\Gamma_2 p_2 = \delta_2 (a_1^2 + a_2),
\]

\[
(A.17.2)
\]

i.e. LHS = RHS.

It can be shown by induction that the above results are valid for arbitrary $\mu_2$, i.e. (A.14) is valid.

This completes the proof of (A.5).

Now, we may write (A.4) as follows:

\[
\begin{bmatrix}
F_{11}T_{11} \\
(F_{21}T_{11} + F_{22}T_{21}) \\
F_{22}T_{22}
\end{bmatrix}
= 
\begin{bmatrix}
T_{11}a_{11} \\
(T_{21}a_{11} + T_{22}a_{21}) \\
T_{22}
\end{bmatrix}
\]

\[
(A.18)
\]

where

\[
\begin{align*}
\bar{F}_{21} & = \begin{bmatrix} F_{22} \end{bmatrix}, \quad \bar{A}_{21} = \begin{bmatrix} A_{21} \end{bmatrix}, \quad \bar{T}_{21} = \begin{bmatrix} T_{21} \end{bmatrix}, \\
\bar{F}_{22} & = \begin{bmatrix} F_{22} \ 0 \end{bmatrix}, \quad \bar{A}_{22} = \begin{bmatrix} A_{22} & 0 \end{bmatrix}, \quad \bar{T}_{22} = \begin{bmatrix} T_{22} & 0 \end{bmatrix}.
\end{align*}
\]
In the previous treatment we proved that
\[ \tilde{F}_{22}\tilde{T}_{22} = \tilde{T}_{22}\tilde{A}_{22}. \]

We will have to prove, further, that
\[ \tilde{F}_{21}\tilde{T}_{11} + \tilde{F}_{22}\tilde{T}_{21} = \tilde{T}_{21}\tilde{A}_{11} + \tilde{T}_{22}\tilde{A}_{21}. \]  
(A.19)

But (A.19) has the same form as (A.5) and, therefore, the same procedure which led to the proof of (A.5) can be used to prove (A.19). Doing so, we arrive at the proposed expression for evaluating \( \delta_1 \), namely,
\[ \delta_1 = \tilde{F}_{22}^{-1}\tilde{T}_{22}\beta_1. \]  
(A.20)

The matrices to be inverted in (A.15) and (A.20) are block triangular with the diagonal blocks in the Mansour form.

The inverse of the Mansour matrix can be directly computed as follows:
For \( F \) and \( n \times n \) matrix of the form (2),
\[
\begin{bmatrix}
-\Delta_1 & -(1-\Delta_1^2)\Delta_2 & \cdots & (1-\Delta_1^2)(1-\Delta_2^2)\cdots(1-\Delta_{j-1}^2)\Delta_j & \cdots & (1-\Delta_1^2)(1-\Delta_2^2)\cdots(1-\Delta_{n-1}^2)\Delta_n \\
1 & -\Delta_1\Delta_2 & \cdots & (1-\Delta_1^2)(1-\Delta_2^2)\cdots(1-\Delta_{j-1}^2)\Delta_j & \cdots & (1-\Delta_1^2)(1-\Delta_2^2)\cdots(1-\Delta_{n-1}^2)\Delta_n \\
0 & 1 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

\( i = 1, 2, 3, \ldots, n, j = 1, 2, 3, \ldots, n, \Delta_0 = 1. \)

It can also be shown that
\[
F^{-1} = DF^*D^{-1}
\]

where
\[
D = \text{diag} \left[ \prod_{i=1}^{n} (1-\Delta_i^2), \ldots, \prod_{i=j}^{n-1} (1-\Delta_i^2), \ldots, 1 \right] \]  
(A.22)

and \( F^* = F^T \) with \( \Delta_n \) replaced by \( 1/\Delta_n \) and \( T \) means transpose.

In the same manner, the proof for similarity of the pairs \( (A, B) \) and \( (F, G) \) could be established for an arbitrary number of diagonal blocks, when one starts by the lowest block and proceeds upwards.

5. References


