1 Lyapunov Stability Theory

In this section we consider proofs of stability of equilibria of autonomous systems. This is standard theory for nonlinear systems, and one of the most important tools in the analysis of nonlinear systems. It may be extended relatively easily to cover non-autonomous systems and to provide a strategy for constructing a stabilizing feedback controller.

In the sequel we present the results for time invariant systems. They may be derived for time varying systems as well, but the essential idea is more accessible for the time invariant case.

1.1 Autonomous Systems & Stability

Consider the autonomous system

$$\dot{x} = f(x)$$

where \( f : D \rightarrow \mathbb{R}^n \) is a locally Lipschitz map from a domain \( D \subset \mathbb{R}^n \) into \( \mathbb{R}^n \). Suppose \( \bar{x} \in D \) is an equilibrium point of the system; that is \( f(\bar{x}) = 0 \). Our goal is to characterize and study the stability of \( \bar{x} \). For convenience, we state all definitions and theorems for the case when the equilibrium point is at the origin of \( \mathbb{R}^n \); that is, \( \bar{x} = 0 \). There is no loss of generality in doing so because any equilibrium point can be shifted to the origin via a change of variables. Suppose \( \bar{x} \neq 0 \) and consider the change of variables \( y = x - \bar{x} \). The derivative of \( y \) is given by

$$\dot{y} = \dot{x} = f(x + \bar{x}) \equiv g(y)$$

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \forall t \geq 0$$

In the new variable \( y \), the system has equilibrium at the origin. Therefore, without loss of generality, we will assume that \( f(x) \) satisfies \( f(0) = 0 \) and study the stability of the origin \( x = 0 \).

**Definition 1.1:** The equilibrium point \( x = 0 \) of \( \dot{x} = f(x) \) is

1. stable if, for each \( \varepsilon > 0 \), there is \( \delta = \delta(\varepsilon) > 0 \) such that

2. Unstable if it is not stable

3. Asymptotically stable if it is stable and \( \delta \) can be chosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$$
**Definition 1.2:** The system $\dot{x} = f(x)$ is called (locally) exponentially asymptotically stable about $x = 0$ if

1. $\dot{x} = f(x)$ is asymptotically stable about $x = 0$
2. $\exists \delta > 0, M > 0, \lambda > 0$ such that $\left\| x(t_0) \right\| < \delta \Rightarrow \forall t > t_0 \left\| x(t) \right\| \leq M e^{-\lambda(t-t_0)} \left\| x(t_0) \right\|$

1.2 Lyapunov’s 1st or Direct Method

Let $V:D \to \mathbb{R}$ be a continuously differentiable function defined in a domain $D \subset \mathbb{R}^n$ that contains the origin. The rate of change of $V$ along the trajectories of $\dot{x} = f(x)$ denoted by $\dot{V}(x)$, is given by

$$\dot{V}(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) = \frac{\partial V}{\partial x} f(x)$$

This is called the **total or absolute derivative** of $V$.

The absolute derivative of $V$ along the trajectories of a system is dependent on the system’s equation. Hence, $\dot{V}(x)$ will be different for different systems. If $\phi(t,x)$ is the solution of $\dot{x} = f(x)$ that starts at initial state $x$ at time $t = 0$, then

$$\dot{V}(x) = \left. \frac{d}{dt} V(\phi(t,x)) \right|_{t=0}$$

**Example 1.1:** Consider the system

$$f(x) = \begin{pmatrix} -x_1 + 2x_1^2x_2 \\ -x_2 \end{pmatrix} = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

With the candidate Lyapunov function:

$$V(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2$$

Calculate the total derivative of $V$:

$$\nabla V(x) = [2\lambda_1 x_1, 2\lambda_2 x_2]$$

then

$$\dot{V}(x) = 2\lambda_1 x_1 (-x_1 + 2x_1^2x_2) + 2\lambda_2 x_2 (-x_2) = -2\lambda_1 x_1^2 - 2\lambda_2 x_2^2 + 4\lambda_1 x_1^3 x_2$$
Therefore, if $\dot{V}(x)$ is negative, $V$ will decrease along the solution of $\dot{x} = f(x)$. We are now nearly ready to state Lyapunov’s stability theorem. The following definition will be necessary

**Definition 1.3:** A domain $D \subseteq \mathbb{R}^n$ is called invariant for the system $\dot{x} = f(x)$ if $x(t_0) \in D \Rightarrow x(t) \in D \ \forall t \geq t_0$.

**Theorem 1.1: (Lyapunov’s First or Direct Method)**

Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$ and $D \subseteq \mathbb{R}^n$ be invariant, and let $V : D \to \mathbb{R}$ be a continuously differentiable function such that

1. $V(0) = 0$ and $V(x) > 0 \ \forall x \in D \setminus \{0\}$
2. $\dot{V}(x) \leq 0 \ \forall x \in D$

Then, $x = 0$ is stable. Moreover, if

3. $\dot{V}(x) < 0 \ \forall x \in D \setminus \{0\}$

then $x = 0$ is asymptotically stable.

**Remarks**

1. If (1) above holds, then $V$ is called *locally positive definite* – lpd. If only $V(x) \geq 0, \ \forall x \in D \setminus \{0\}$, then $V$ is *locally semi-positive definite*.
2. If (1) and (2) hold, the $V$ is called a *Lyapunov Function* for the system $\dot{x} = f(x)$.

**Proof of Theorem 5.1**

Proof of stability: We consider the level sets of the Lyapunov function.

Let $\varepsilon > 0$, choose $r \in (0, \varepsilon]$ such that $B_r = \left\{ x \in \mathbb{R}^n, \|x\| \leq r \right\} \subset D$. Let $\alpha = \min_{\|x\|} V(x)$. Choose $\beta = (0, \alpha)$. Define $\Omega_\beta = \{ x \in B_r, V(x) \leq \beta \}$. 

It holds that if \( x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \ \forall t \) because
\[
\dot{V}(x(t)) < 0 \Rightarrow V(x(t)) \leq V(x(0)) \leq \beta
\]
Further \( \exists \delta > 0 \) such that \( \|x\| < \delta \Rightarrow V(x) < \beta \)
\[
B_\delta \subseteq \Omega_\beta \subseteq B_r
\]
\[
x(0) \in B_\delta \Rightarrow x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \Rightarrow x(t) \in B,
\]
\[
\|x(0)\| < \delta \Rightarrow \|x(t)\| \leq \epsilon \quad \forall t > 0
\]

Proof of asymptotic stability:
We construct a proof by contradiction. Let \( v(t) = V(x(t)) \), but assume that \( x(t) \rightarrow x^* \neq 0 \). Convergence of \( x(t) \) to a fixed point implies that \( \dot{V}(t) \rightarrow 0 \). But by condition (3) \( \dot{v}(x^*) = v(t \rightarrow \infty) < 0 \). This is a contradiction. Thus \( x^* = 0 \), showing that all trajectories converge to the origin, thus the system is asymptotically stable.

Example 1.1 (ctd)
Recall that
\[
\dot{V}(x) = -2\lambda_1 x_1^2 - 2\lambda_2 x_2^2 + 4\lambda_1 x_1^3 x_2
\]
Consider that \( \lambda_1 = \lambda_2 = 1 \). Then the total derivative is given by
\[
\dot{V}(x) = -2x_1^2 - 2x_2^2(1 - 2x_1 x_2)
\]
\[
g(x) = -2x_2^2 - 2x_1^2 g(x)
\]
Then the total derivative is guaranteed to be negative whenever \( g(x) > 0 \).
The level sets of $V$, where $\dot{V} < 0$ will be invariant. Thus the red circle above is level, and within this circle, $\dot{V} < 0$, so we conclude that the origin is locally asymptotically stable.

**Theorem 1.2: (Instability result)** Suppose $V(0) = 0$, $V \in C^1(D)$ and that $\forall \delta > 0$ $\exists x_0 \neq 0$, $\|x_0\| < \delta$ such that $V(x_0) > 0$. Let $r > 0$ such that $B_r \subset D$ and $U = \{x \in B_r \mid V(x) > 0\}$. Suppose $\dot{V}(x) > 0$ in $U$. Then $x = 0$ is unstable.

**Proof**
Note that the set $U = \{x \in B_r \mid V(x) > 0\}$ is connected due to continuity of $V$. Furthermore, the origin must be on the border of $U$, as there exists elements of $U$ which are arbitrarily close to 0. Note that as $\dot{V}(x) > 0$ for $x \in U$, trajectories starting in $U$ will not leave $U$ on the border defined by $V(x) = 0$, rather they will leave via $B_r$. Thus there are trajectories starting arbitrarily close to the origin that will leave a ball of a given radius, proving instability of the point.

Note that the requirements on $V(x)$ are not as strict on the requirements on a Lyapunov function.

**Example 1.2:** The set $U$ for $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$ is shown in the following diagram.

Consider the system $\dot{x}_1 = x_1$, $\dot{x}_2 = -x_2$, then $\dot{V}(x) > 0$ for $x \in U$, proving that the system is unstable.
Example 1.3: Mass-Spring System
Consider a mass $m$ on a spring exerting a force $F_d$ and subject to a frictional force $F_v$ when in motion, as given by:

$$F_v = -\delta \dot{x} \text{ and } F_d = -F(x)$$

Where $x = 0$ is defined as the equilibrium, the point where there is no force exerted by the spring.

Then the system becomes

$$m \ddot{x} = -F(x) - \delta \dot{x}$$

and with $x = x_1$, $\dot{x} = x_2$ and $m = 1$ we obtain

$$\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = -F(x_1) - \delta x_2
\end{cases} = f(x)$$

Consider the Lyapunov function $V(x) = \int_0^x F(s) \, ds + \frac{1}{2} x_2^2$

Then $\nabla V(x) = [F(x_1), x_2]$ and $\dot{V}(x) = F(x_1) x_2 + x_2 (-F(x_1) - \delta x_2) = -\delta x_2^2$

The system is stable but we can’t prove asymptotically stability because $V \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = 0 \ \forall x_1$.

In order to prove stability we need a more general result – LaSalle’s Invariance Principle.
1.3 The Invariance Principle

Theorem 1.3: (Lasalle’s Invariance Principle)
Let $\Omega \subset D$ be a compact set that is positively invariant with respect to $\dot{x} = f(x)$. Let $V : D \rightarrow \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in $\Omega$. Let $E$ be the set of all points in $\Omega$ where $\dot{V}(x) = 0$. Let $M$ be the largest invariant set in $E$. Then every solution starting in $\Omega$ approaches $M$ as $t \rightarrow \infty$.

Example 1.3 (ctd): Mass-Spring System

Applying this notation to the example above, we obtain

$E = \{ (x_1, x_2) : x_2 = 0 \}$

$M = \{ 0 \}$

Thus, trajectories must converge to the origin, and we have proven that the system is asymptotically stable.

Remark: La Salle’s result can also be used to find limit cycles.
1.4 Linear Systems and Linearization

In this section we prove stability of the system by considering the properties of the linearization of the system. Before proving the main result, we require an intermediate result.

**Definition 1.4:** A matrix $A$ is called Hurwitz if $\Re(\lambda) > 0$.

Consider the system $\dot{x} = Ax$. We look for a quadratic function

$$V(x) = x^TPx$$

where $P = P^T > 0$. Then

$$\dot{V}(x) = \dot{x}^TPx + \dot{x}^TP\dot{x} = x^T(A^TP + PA)x = -x^TQx$$

If there exists $Q = Q^T > 0$ such that

$$A^TP + PA = -Q,$$

then $V$ is a Lyapunov function and $x = 0$ is globally stable. This equation is called the *Matrix Lyapunov Equation*.

We formulate this as a matrix problem: Given $Q$ positive definite, symmetric, how can we find out if there exists $P = P^T > 0$ satisfying the Matrix Lyapunov equation.

The following result gives existence of a solution to the Lyapunov matrix equation for any given $Q$.

**Theorem 1.4:** For $A \in \mathbb{R}^{n \times n}$ the following statements are equivalent:

1) $A$ has all eigenvalues left of the j-axis
2) For all $Q = Q^T > 0$ there exists $P = P^T > 0$ satisfying $A^TP + PA = -Q$.

**Proof outline**

We make a constructive proof. For a given $Q = Q^T > 0$, consider the following candidate solution for $P$:

$$P = \int_0^\infty e^{\lambda t}Qe^{\lambda t}dt$$

That $P = P^T > 0$ follows from the properties of $Q$. Note that the integral will converge if and only if $A$ is a Hurwitz matrix. We now show that $P$ satisfies the matrix Lyapunov equation:
This theorem has an interesting interpretation in terms of the energy available to a system available. If we say that the energy dissipated at a particular point in phase space $x$ is given by $q(x) = x^T Q x$ - meaning that a trajectory passing through $x$ is loosing $q(x)$ units of energy per unit time, then the equation $V(x) = x^T P x$, where $P$ satisfies the matrix Lyapunov equation gives the total amount of energy that the system will dissipate before reaching the origin. Thus $V(x) = x^T P x$ measures the energy stored in the state $x$.

With this result we are in a position to prove the following result:

**Theorem 1.5: (Lyapunov’s Second or Indirect Method)**

Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$ where $f : D \to \mathbb{R}^n$ is a continuously differentiable and $D$ is a neighborhood of the origin. Let

$$A = \frac{\partial f}{\partial x}_{x=0}$$

Then

1. The origin is asymptotically stable if $\text{Re} (\lambda_i) < 0$ for all eigenvalues of $A$
2. The origin is unstable if $\text{Re} (\lambda_i) > 0$ for one or more of the eigenvalues of $A$

**Proof**

If $A$ is Hurwitz, then there exists $P = P^T > 0$ so that $V(x) = x^T P x$ is a Lyapunov function of the linearized system.

Then $f(x) = Ax + g(x)$ where $\frac{\|g(x)\|}{\|x\|} \to 0$ for $\|x\| \to 0$

Then $V(x) = f(x)^T P x + x^T P f(x) = x^T (P A + A^T P) x + 2 x^T P g(x) = \underbrace{-x^T Q x}_{\leq 0} + \underbrace{2 x^T P g(x)}_{\leq 0}$
As we converge to 0, we enter the region where $\|x^T Q x\| >> 2x^T P g(x)$, yielding $\dot{V}(x) < 0$, and so $x = 0$ is locally asymptotically stable. This proves point 1 of the theorem.

To prove point 2: Consider that $\Re(\lambda_i) > 0$ for some $i$. For simplicity consider $\lambda_i$ real, and let $v$ be the associated eigenvector. Then locally a trajectory starting at $x(0) = \varepsilon v$ for sufficiently small $\varepsilon$ will be given by $x(0) = \varepsilon e^{\lambda_i v}$, so that it grows exponentially. Thus the origin must be unstable.

For $\lambda_i$ complex, consider the associated complex conjugate eigenvalues, and the associated eigenvectors. Locally this also forms an exponentially growing trajectory, showing instability of the origin.

Note. The theorem does not say anything when $\Re(\lambda_i) \leq 0$ \forall $i$ with $\Re(\lambda_i) = 0$ for some $i$. In this case linearization fails to determine the stability of the equilibrium point, and further analysis is necessary. This is illustrated by the following example:

**Example 1.4:**

1. $\dot{x} = ax^3$, $a > 0$ \hspace{1cm} $x = 0$ is unstable
2. $\dot{x} = -ax^3$, $a > 0$ \hspace{1cm} $x = 0$ is asymptotically stable

In both cases the matrix $A$ of the linearized system is the same: $\dot{x} = 0$

The multi-dimensional result which is relevant here is the Center Manifold Theorem. This theorem is beyond the scope of this course, however the result is approximately that around any equilibrium point of a nonlinear system, the space may be split into:

1. An exponentially unstable manifold: An invariant set corresponding to the eigenvalues of the linearized system with positive real part
2. An exponentially stable manifold: An invariant set corresponding to the eigenvalues of the linearized system with negative real part
3. The Center Manifold: An invariant set corresponding to the eigenvalues of the linearized system with negative real part, which may contain asymptotically stable or unstable sub-manifolds, but exhibiting no exponential convergence or divergence.
1.5 Converse Theorems

Consider the following problem:

Let \( x = 0 \) be asymptotically stable.
Does there exist a Lyapunov function for the system?

The closest answer to this question may be provided by the following system:

**Theorem 1.6:** Let \( x = 0 \) be locally exponentially asymptotically stable for the system \( \dot{x} = f(x) \). Then there exists a Lyapunov function \( V : D \to \) for the system such that:

1) \( C_1 \| x \|^2 \leq V(x) \leq C_2 \| x \|^2 \)
2) \( \dot{V}(x) \leq -C_3 \| x \|^2 \)
3) \( \| \nabla V(x) \| \leq C_4 \| x \| \)

where \( C_1, C_2, C_3, C_4 > 0 \)

**Proof Idea**
Analogously to the construction of a solution to the matrix Lyapunov equation, a Lyapunov Function is constructed by:

\[
V(x) = \int_0^\infty \phi(t, x)^T \phi(t, x) dt
\]

Where \( \phi(t, x) \) is the solution to the system differential equations, defining the trajectory starting at \( x \) at time \( t=0 \). Due to local exponential stability, the integral may be shown to converge locally. By bounding the rate of growth of the integral away from 0, the properties may be proven. See Khalil, 3rd edition, Theorem 3.12, p 149-152

Determining a Lyapunov function may also be expressed as the result of solving a partial differential equation:

Consider a locally positive definite function \( q(x) \). Then if there exists a solution \( V(x) \) to the partial differential equation given by:

\[
V(0) = 0
\]
\[\nabla V(x) = -q(x)\]

Then \( V(x) \) is a Lyapunov function for the system, and the equilibrium must be stable.