1 \( H_\infty \) Control.

1.1 Introduction

1.1.1 The Linear Quadratic Regulator (LQR)

Consider the system
\[
\dot{x} = Ax + Bu
\]

And the performance criteria
\[
J[u(\cdot)] = \int_0^\infty [x^TQx + u^TRu]dt, \quad Q \geq 0, \ R > 0,
\]

**Problem:** Calculate function \( u : [0, \infty) \mapsto \mathbb{R}^n \) such that \( J[u] \) is minimized.

**Remarks:**

1. LQR can be considered for final times
2. LQR can be considered for time varying matrices
3. LQR can be extended in several ways to nonlinear systems (e.g. State Dependent Riccati Equations)
4. LQR assumes full knowledge of the state

The LQR controller has the following form
\[
u(t) = -R^{-1}B^TPx(t)
\]

Where \( P \in \mathbb{R}^{n \times n} \) is given by the positive (symmetric) semi definite solution of

\[
0 = PA + A^TP + Q - PBR^{-1}B^TP
\]

This equation is called Riccati equation. It is solvable iff the pair \((A, B)\) is controllable and \((Q, A)\) is detectable

**LQR controller design**

1. \((A, B)\) is given by “design” and cannot be modified at this stage
2. \((Q, R)\) are the controller design parameters. Large \( Q \) penalizes transients of \( x \), large \( R \) penalizes usage of control action \( u \).
1.1.2 The Linear Quadratic Gaussian Regulator (LQG)

In LQR we assumed that the whole state is available for control at all times (see formula for control action above). This is unrealistic as the very least there is always measurement noise.

One possible generalization is to look at

\[ \dot{x} = Ax + Bu + w \]
\[ y = Cx + v \]

Where \( v, w \) are stochastic processes called measurement and process noise respectively. For simplicity one assumes these processes to be white noise (ie zero mean, uncorrelated, Gaussian distribution).

Now only \( y(t) \) is available for control. It turns out that for linear systems a separation principle holds

1. First, calculate \( \hat{x}(t) \) estimate the full state \( x(t) \) using the available information
2. Secondly, apply the LQR controller, using the estimation \( \hat{x}(t) \) in place of the true (now unknown) state \( x(t) \).

**Observer design (Kalman Filter)**

The estimation \( \hat{x}(t) \) is calculated by integrating in real time the following ODE

\[ \dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \]

With the following matrices calculated offline (ie before hand)

\[ L = PC^T R^{-1} \]
\[ 0 = AP + PA^T - PC^T R^{-1} CP + Q, P \geq 0, \]
\[ Q = E(ww^T), R = E(vv^T) \]

The Riccati equation above has its origin in the minimization of the cost functional

\[ J[\hat{x}(\cdot)] = \int_{-\infty}^{0} [(\hat{x} - x)(\hat{x} - x)^T]dt \]

1.2 \( H_\infty \) Control

Let us introduce the so called \( H_\infty \) norm of a transfer function \( G(s) \in \mathbb{C}^{p \times q} \). This is a mapping from the space of matrix transfer functions into the positive real numbers defined by
\[ \|G(s)\|_{\infty} = \max_{\omega \in \mathbb{R}} \|G(j\omega)\| = \max_{\omega \in \mathbb{R}} \|G(j\omega)\|^2 = \max_{\omega \in \mathbb{R}} \sigma_1(G(j\omega)), \]

where \( \sigma_1 \) denotes the largest singular value of the complex matrix \( G(j\omega) \).

The intuition associated to this definition is that the \( H_\infty \) norm quantifies the maximal amplification that a signal may have once applied to the system given by the transfer matrix \( G(s) \).

![Figure 1. \( H_\infty \) norm](image)

\[ \|F\|_{\infty} = \sup \frac{\|Fu\|_{L^2}}{\|u\|_{L^2}} \]

**Figure 1. \( H_\infty \) norm**

Let us now consider the system depicted below.

![Figure 2. \( H_\infty \) Objects](image)

**Figure 2. \( H_\infty \) Objects**

Here \( w(t) \) is a disturbance acting on the system, while \( u(t) \) is the control action. Further, \( z(t) \) is to be understood as performance index. Without loss of generality, one assumes that this continuous-time system is given by the matrices and equations below.
\[ \dot{x} = Ax + B_1w + B_2u \]
\[ z = C_1x + D_{12}w \]
\[ y = C_2x + D_{21}w \]

**Figure 3. Underlying H_\infty System Equations**

**Problem:** H_\infty Control is about finding the stabilizing control (not necessarily memoryless) law \( u(t) = F(y(t)) \) that stabilizes the system above AND minimizes the effect of the disturbance \( w(t) \) on the performance index \( z \).

This goal can be achieved by minimizing the H_\infty norm of the transfer function \( T: w \mapsto z \) that is generated once a controller design has been chosen.

**GOAL**
\[ z(s) = \Phi(s)w(s) \]

- **Influence from Disturbance on performance variable**
- **Make this as small as possible**

**Figure 4. H_\infty Goal**

In other words, one designs a controller that minimizes the effect of the worst possible disturbance.

Another related formulation is to have a controller that renders the system dissipative and internally stable, see below.

- **Dissipative**
\[ \int_0^t z^2 ds \leq \gamma^2 \int_0^t w^2 ds + \beta(x_0) \]
- **Internally stable**
\[ w \in L_2 \Rightarrow x, u, z, y \in L_2 \]

**1.2.1 Linear H_\infty Control Design**

By analogy with the LQG case, for the design of this controller we expect to have to solve 2 Riccati equations, one for calculation the optimal action given the system state, and one for generating an estimation of the state.
The main difference now is that we are dealing now with two objects

1. The worst possible disturbance
2. The best possible reaction to that worst disturbance

Relation to the Game Theory is seen here, where we have two players, one that is trying to maximize a cost function and one that is trying to minimize it. The applicable cost function here is

\[ J[w, u, x0] = \int_0^t (z^2 - \gamma w^2)ds \]

subject to the equations, see Figure 3.

In practice one does not seek the optimal controller (i.e. the ones that produces the absolute minimal value for \( \|P\|_\infty \) but the design is made using an iterative procedure that seeks for reducing the norm while still looking at other performance measures. Indeed, for any sufficiently large given \( \gamma > 0 \) one can calculate a controller that makes \( \|P\|_\infty < \gamma \) using the following formulae

i) Find \( X \geq 0 \) that satisfies the ARE

\[ 0 - XA + A^TX - C_1^TC_1 + X(1/\gamma^2B_1B_1^T - B_2B_2^T)X \]

And

\[ A - (1/\gamma^2B_1B_1^T - B_2B_2^T)X \text{ is stable} \]

ii) Find \( Y \geq 0 \) that satisfies the ARE

\[ 0 - YA + A^TY - B_1B_1^T + Y(1/\gamma^2C_1^TC_1 - C_2^TC_2)Y \]

And

\[ A - (1/\gamma^2C_1^TC_1 - C_2^TC_2)Y \text{ is stable} \]

iii) \( \rho(XY) < \gamma^2 \) (lowest singular value)

Then the dynamic H\( \infty \) controller has the form

\[ \dot{x} = Ax + B_1w_{\text{worst}}(t) + B_2u_{\text{opt}}(t) + ZL(C_2\hat{x}(t) - y(t)) \]

\[ w_{\text{worst}}(t) = \gamma^2B_1^TX\hat{x}(t) \]

\[ u_{\text{opt}}(t) = F\hat{x}(t) \]

where

\[ Z = (I - 1/\gamma^2XY)^{-1} \]

\[ L = -YC_2^T \]

\[ F = -B_2X \]
Remarks.
1. For very small values of $\gamma$, these equations will have no solution while for very large $\gamma$, these equations reduce to the LQG case.
2. The best design is made by an iterative procedure reducing $\gamma$ that keeps an eye on the disturbance rejection properties at all interesting frequencies (and not just the maximum value).

It can be proved that these Riccati Equations may have solution only if the following “structural” assumptions hold.

- (A1) $(A, B_2, C_2)$ is stabilizable and detectable
- (A2) $D_{12}$ and $D_{21}$ has full rank
- (A3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all $\omega$
- (A4) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all $\omega$

Figure 5. Linear $H_\infty$ Assumptions

These assumptions are rather natural as they transfer properties as observability and controllability to this new context.

Remark. $H_\infty$ design provides robustness at the cost of a pessimistic control law: it assumes that the worst possible perturbation is acting on the system at all times.

1.2.2 Nonlinear $H_\infty$ Control Design

All elements of the linear case are here present, and we expect a solution schema where one has an optimal control law given the full system state, and a law that helps us to generate optimal estimations of this actually unknown full system state.

The difference we do expect though is that in place of 2 Riccati equations, one for optimal control and one for the optimal observer, we will have to deal here with two Hamilton Jacobi equations.

We have met these partial differential equations already, when we treated “Optimal Control and Dynamic Programming”. We know they are hyperbolic, that they are amenable for treatment with numerical methods, and that they have an interesting mathematical theory due to the fact that their solutions may become discontinuous and those usual concepts like differentiability cannot be applied straightforwardly.

The theory has been developed for nonlinear systems of the form.
\[ \dot{x} = A(x)x + B_1(x)w + B_2(x)u \]
\[ z = C_1(x)x + D_{12}(x)w \]
\[ y = C_2(x)x + D_{21}(x)w \]

For this system the “Observation” PDE has the form.

\[
\frac{\partial P}{\partial t} = -\nabla_x P \cdot \left( A + B_1 \frac{D_{21}}{D_{21}} (y - C_2) + B_2 u \right) + \frac{1}{2\gamma^2} \nabla_x P B_1 \left[ I - D_{21} D_{21} \right] B_1' \nabla_x P_e \cdot \nabla_x P_e' + \frac{1}{2} \left| C_1 + D_{12} u \right|^2 - \frac{\gamma^2}{2} (y - C_2)(y - C_2)
\]

**A nonlinear first-order PDE of Hamilton-Jacobi type.**

Figure 6. Observation Hamilton Jacobi Equation

It turns out that the best estimate of the full state is given by the formula.

\[ \bar{x}(p) = \arg \max_x (p(x) + V(x)) \]

where \( V(x) \) is the solution of the optimal control Hamilton Jacobi PDE, and the control law is then given by

\[ u^*(p) = u^*_{state}(\bar{x}(p)) = -C_1(\bar{x}(p)) - B_2(\bar{x}(p)) \nabla_x V(\bar{x}(p))' \]

Naturally, we expect the system matrix functions to have good structural properties for this PDE to make sense in the first place. So we expect

1. \( A(x), C_1(x), C_2(x) \) globally Lipschitz, smooth, vanish at 0
2. \( B_1(x), B_2(x) \) globally Lipschitz, bounded, smooth
3. \( D_{12}(x), D_{21}(x) \) constant (for simplicity)

1.3 References