1 Optimal Control

We consider first simple unconstrained optimization problems, and then demonstrate how this may be generalized to handle constrained optimization problems. With the observation that an optimal control problem is a form of constrained optimization problem, variational methods are used to derive an optimal controller, which embodies Pontryagin’s Minimum Principle. Subsequently an alternative approach, based on Bellman’s Principle of Optimality, and Dynamic programming is used to derive the Hamilton-Jacobi equations.

1.1 Unconstrained Optimization

Consider a function

\[ L : \mathbb{R} \rightarrow \mathbb{R} \]

We want to find

\[ \min_u L(u) \]

Let us assume that \( L \) is sufficiently smooth, and consider the Taylor expansion:

\[ L(u) = L(u_0) + \frac{dL}{du}|_{u=u_0}(u - u_0) + \frac{dL^2}{du^2}|_{u=u_0}(u - u_0)^2 + .. \]

Then we have a necessary condition

\[ \frac{dL}{du}|_{u=u_0} = 0 \]

and a sufficient condition

\[ \frac{dL^2}{du^2}|_{u=u_0} > 0 \]

Note that these are only conditions for a local minimum. Additional conditions are required to find the global minimum if the function is non-convex. If we have a function with more than one variable, that is \( L : \mathbb{R}^n \rightarrow \mathbb{R} \) we have the following conditions

\[ \left\{ \frac{\partial L}{\partial u_1}, \frac{\partial L}{\partial u_2}, ..., \frac{\partial L}{\partial u_n} \right\}|_{u=u_0} = 0 \]
and

\[
\frac{\partial^2 L}{\partial u^2}|_{u=u_0} = \begin{pmatrix}
\frac{\partial^2 L}{\partial u_1^2} & \cdots & \frac{\partial^2 L}{\partial u_1 \partial u_n} \\
\cdots & \cdots & \cdots \\
\frac{\partial^2 L}{\partial u_n \partial u_1} & \cdots & \frac{\partial^2 L}{\partial u_n^2}
\end{pmatrix}
|_{u=u_0} > 0
\]

i.e. is positive definite.

**Example 1** Find the minimum of the function

\[f(x_1, x_2) = 6x_1^2 + 2x_2^2 + 3x_1x_2 - 8x_1 + 3x_2 - 4.\]

**Necessary condition**

\[
\frac{\partial f}{\partial u} = \begin{pmatrix} 12x_1 + 3x_2 - 8 \\ 4x_2 + 3x_1 + 3 \end{pmatrix} = 0
\]

Solving these equations we find \(x_0 = \left( \frac{41}{30}, -\frac{20}{13} \right)\) and when we insert \(x_0\) into the Hessian Matrix we see that

\[
\frac{\partial^2 L}{\partial u^2}(x_0) = \begin{pmatrix} 12 & 3 \\ 3 & 4 \end{pmatrix}.
\]

the resulting matrix is positive definite. We conclude that the point \(x_0\) is a minimum.

The plot in Figure 1 shows the function for which we are finding the minimum.

**1.2 Constrained Optimization**

**Theorem 1** Consider the problem

\[\min_x L(x), \quad f(x) = 0\]

with \(L : \mathbb{R}^n \to \mathbb{R}\) and \(f : \mathbb{R}^n \to \mathbb{R}^m\). Then this problem is equivalent to

\[\min_{x, \lambda} L(x) + \lambda f(x).\]
Figure 1: Convex Function
The function

\[ H(x, \lambda) = L(x) + \lambda f(x) \]

is called the **Hamiltonian** of the optimization problem. The coefficients \( \lambda \in \mathbb{R}^m \) are called **Lagrange Multipliers** of the system.

**Proof**

Without loss of generality we consider \( x \in \mathbb{R}^2 \). The necessary conditions for a minimum are

\[
\frac{\partial H}{\partial x, \lambda} = 0
\]

or

\[
\begin{align*}
\frac{\partial H}{\partial x_1} &= \frac{\partial L}{\partial x_1} + \lambda \frac{\partial f}{\partial x_1} \\
\frac{\partial H}{\partial x_2} &= \frac{\partial L}{\partial x_2} + \lambda \frac{\partial f}{\partial x_2} \\
\frac{\partial H}{\partial \lambda} &= f(x_1, x_2)
\end{align*}
\]

The third condition is equivalent to the boundary conditions of the original problem being satisfied. The first two conditions are equivalent to saying that the vectors

\[
\left(\begin{array}{c}
\partial L \\
\partial f
\end{array}\right)_{\partial x_1} ; \left(\begin{array}{c}
\partial L \\
\partial f
\end{array}\right)_{\partial x_2}
\]

are parallel or colinear. If these vectors are parallel, then the matrix

\[
\left(\begin{array}{cc}
\frac{\partial L}{\partial x_1} & \frac{\partial f}{\partial x_1} \\
\frac{\partial L}{\partial x_2} & \frac{\partial f}{\partial x_2}
\end{array}\right)
\]

has rank less than 2, which means that the linear system obtained by equating to zero the derivative of the Hamiltonian has a non trivial solution on \( \lambda \).

With the help of a diagram in Figure 2, it is easy to understand that where we have a minimum or maximum the two gradients (with the red vector representing the gradient of \( L \) and the black vector representing the gradient of \( f \)) have to be parallel, as otherwise one can increase or decrease the value of \( L \) while satisfying the constraint \( f(x) = 0 \).
Figure 2: Gradients of $L$ and $f$ must be colinear at the extrema
Example 2  Find the minimum of the function

\[ L(x_1, x_2) = x_1^2 - 4x_1 + x_2^2 \]

with the constraints

\[ x_1 + x_2 = 0. \]

Indeed, the Hamiltonian is given by

\[ H(x, \lambda) = x_1^2 - 4x_1 + x_2^2 + \lambda(x_1 + x_2). \]

Thus, the expressions for

\[ \frac{\partial H}{\partial x, \lambda} = 0 \]

are

\[ 0 = \frac{\partial H}{\partial x_1} = 2x_1 - 4 + \lambda \]

\[ 0 = \frac{\partial H}{\partial x_2} = 2x_2 + \lambda \]

\[ 0 = \frac{\partial H}{\partial \lambda} = x_1 + x_2 \]

Solving this system of linear equations we find the solution \((x_1, x_2, \lambda) = (1, -1, 2)\). This solution is only a candidate for solution. One must now test whether these coordinates represented the solution we seek.

Example 3  Consider the problem

\[ \min_{x_1, x_2, x_3} L(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2, \]

such that

\[ x_1x_2 + x_2x_3 + x_3x_1 = 1. \]

As per "recipe" we write the Hamiltonian.

\[ H(x, \lambda) = x_1^2 + x_2^2 + x_3^2 + \lambda(x_1x_2 + x_2x_3 + x_3x_1 - 1). \]

Taking the derivatives we see

\[ 0 = 2x_1 + \lambda x_2 + \lambda x_3 \]

\[ 0 = 2x_2 + \lambda x_1 + \lambda x_3 \]

\[ 0 = 2x_3 + \lambda x_1 + \lambda x_2 \]

\[ 0 = x_1x_2 + x_2x_3 + x_3x_1 - 1 \]
From the first three equations we see that
\[ x = y = z. \]
This result plus the last equation tell us that
\[ x = y = z = \sqrt{\frac{1}{3}}, \quad \lambda = -1. \]
Again, we must make sure that this candidate is indeed a solution to our original problem.

### 1.3 Pontryagin’s Minimum Principle

Consider the system
\[ \dot{x} = f(x, u), \quad x(0) = x_0 \quad (1) \]
with associated performance index
\[ J[x_0, u(\cdot)] = \phi(x(T)) + \int_0^T L(x(t), u(t))dt \quad (2) \]
and final state constraint
\[ \psi(x(T)) = 0 \quad (3) \]
The following terminology is customary:

1. \( J[x_0, u(\cdot)] \) is called cost function.
2. \( \phi(x(T)) \) is called end constraint penalty
3. \( L(x, u) \) is called running cost
4. \( H(x, u, \lambda) = L(x, u) + \lambda f(x, u) \) is called Hamiltonian.

The **Optimal Control Problem** is: Find the control function
\[ u : [0, T] \mapsto \mathbb{R}^m \]
such that the performance index is minimized and the final state constraint and the system equations are satisfied.
**Theorem 2** Solutions of the Optimal Control Problem also solve the following set of differential equations:

State Equation: \[ \dot{x} = H_\lambda = f(x) \]  

(4)

Co-State Equation: \[ -\dot{\lambda} = H_x = \frac{\partial L}{\partial x} + \lambda \frac{\partial f}{\partial x} \]  

(5)

Optimality Condition: \[ 0 = H_u = \frac{\partial L}{\partial u} + \lambda \frac{\partial f}{\partial u} \]  

(6)

State initial condition: \[ x(0) = x_0 \]  

(7)

Co-state final condition: \[ \lambda(T) = (\phi_x + \psi_x \nu)|_{x(T)} \]  

(8)

where \( \nu \) is the Lagrange multiplier corresponding to end condition given by Equation 3.

**Proof** We use the Lagrange multipliers to eliminate the constraints. Since the main constraints are now given by a dynamical system \( \dot{x} = f(x, u) \) it is clear that they must hold for all \( t \) and thus the vector function \( \lambda : [0, T] \rightarrow \mathbb{R}^n \) is a function of time.

Using the notation \( H(x, u, \lambda) = L(x, u) + \lambda f(x, u) \), the unconstrained minimization problem can be written as

\[
J(x, u, \lambda) = \phi(x(T)) + \psi(x(T))\nu + \int_0^T [L(x, u) + \lambda(f(x, u) - \dot{x})]dt
\]

\[
= \phi(x(T)) + \psi(x(T))\nu + \int_0^T [H(x, u, \lambda) - \lambda \dot{x}]dt.
\]

The differentials are written as

\[
\delta J = [\phi_x(x) + \psi_x(x)\nu]\delta x]|_{x=x(T)} + \\
\int_0^T [H_x \delta x + H_u \delta u + H_\lambda \delta \lambda - \lambda \delta \dot{x} + \delta \lambda \dot{x}]dt + \\
\psi(x)|_{x(T)}\delta \nu
\]

Note that by integrating by parts:

\[
- \int_0^T \lambda \delta \dot{x} = -\lambda \delta x|_{t=T} + \lambda \delta x|_{t=0} + \int_0^T \dot{} \lambda \delta x dt
\]
Furthermore, since $x(0) = x(0)$ is constant, it holds that $\lambda \delta x|_{t=0} = 0$. So we can rewrite the previous expression as

$$
\delta J = \left[ \phi_x(x(t)) + \psi_x(x(t))\nu - \lambda[\delta x]\right]|_{x=x(T)} + \\
\int_0^T \left[ (H_x + \dot{\lambda})\delta x + (H_x - \dot{x})\delta \lambda + H_u \delta u \right] dt + \\
\psi(x)|_{x(T)} \delta \nu
$$

Now for the function $u : [0, T] \to \mathbb{R}^m$ to minimize the cost function, $\delta J$ must be zero for any value of the differentials. Thus, all the expressions before the differentials have to be zero for every $t \in [0, T]$. This observation gives the equations as required.

**Remark 1** Just as in the static optimization case, where the zeros of the derivatives represent candidates to be tested for extremum, the solutions of the system described in Theorem 2 are to be seen as candidates to be the optimal solution and their optimality must be tested for each particular case. In other words, the Pontriaguin Maximum Principle delivers necessary, but not sufficient, conditions for optimality.

**Remark 2** The system of equations in Theorem 2 is a so called ”Two Point Boundary Value” problem. Generally speaking, these problems are only solvable by dedicated software implementing suitable numerical methods.

**Remark 3** The expression regarding $u(\cdot)$ in Equation 6 is a special case of a more general minimum condition. In general we must seek

$$
u^* = \arg\min H(x, u, \lambda)
$$

for given values of $x(t)$ and $u(t)$. In other words, the Pontryagin’s Minimum Principle states that the Hamiltonian is minimized over all admissible $u$ for optimal values of the state and co-state.

**Remark 4** Special attention is needed in the case where $H_u = \text{const}$ for all $u$, in which case the solution is found where the constraints are active. This sort of solution is called “Bang-Bang solution”.
Example 4 Consider the minimization problem

\[ J[x_0, u(\cdot)] = 0.5 \int_0^T u^2(t)dt \]

subject to

\[ \dot{x}_1 = x_2, \quad x_1(0) = x_{10} \]
\[ \dot{x}_2 = u, \quad x_2(0) = x_{20} \]

and with final constraint

\[ \psi(x(T)) = x(T) = 0. \]

Applying Theorem 2, we transform the system into

\[ \dot{x}_1 = H_{\lambda_1} = x_2; \quad x_1(0) = x_{10} \]
\[ \dot{x}_2 = H_{\lambda_2} = u; \quad x_2(0) = x_{20} \]
\[ \dot{\lambda}_1 = -H_{x_1} = 0; \quad \lambda_1(T) = \nu_1 \]
\[ \dot{\lambda}_2 = -H_{x_2} = -\lambda_1; \quad \lambda_2(T) = \nu_2 \]

where

\[ H(x, u, \lambda) = 0.5u^2 + \lambda_1 x_2 + \lambda_2 u \]

Now we see that

\[ H_u = u + \lambda_2 = 0 \]

Thus, we can solve the differential equations and see that

\[ \lambda_1(t) = \nu_1 \]
\[ \lambda_2(t) = -\nu_1(t - T) + \nu_2 \]
\[ u(t) = \nu_1(t - T) - \nu_2 \]

Placing these linear expressions in the dynamic equations for \( x \) and using the initial conditions \( x(0) = x_0 \), we obtain a linear system of equations with respect to \((\nu_1, \nu_2)\), which gives us the final parametrization of the control law \( u \). Figure 3 shows the result of applying this control law with \( T = 15 \).
Figure 3: Trajectories for Minimal Energy Case. Arrival time $T = 15$. 
Example 5 Consider the same problem as in Example 4, but with a performance reflecting minimal time, i.e.

\[ J[x_0, u(\cdot)] = \int_0^T 1 \, dt \]

and constrained input

\[-1 \leq u(t) \leq 1 \quad \forall t \geq 0.\]

Now

\[ H(x, u, \lambda) = 1 + \lambda_1 x_2 + \lambda_2 u. \]

We notice that \( H_u = \lambda_2 \), i.e. \( H_u \) does not depend on \( u \) and thus the extremum is to be reached at the boundaries, i.e. \( u^*(t) = \pm 1, \quad \forall t \geq 0. \) Using

\[ u^* = \arg\min_u H(u) = \arg\min_u \lambda_2(t) u(t) \]

we see that

\[ u^*(t) = -\text{sign} \lambda_2(t). \]

Figure 4 depicts the result of deploying this control. Note that the control law is discontinuous. Further, we observe that the system now needs less time to reach the origin than in the previous "Minimal Energy" example. Of course, this "success" is obtained at the cost of deploying more actuator energy.

1.4 Dynamic Programming

An alternative approach to solving optimal control problems to the variational methods above is given by Dynamic programming, this is based on the following observation.

Remark 5 Bellman’s principle of optimality: An optimal policy (control law) has the property that no matter what the previous decisions (control) have been, the remaining decisions must constitute an optimal policy with respect to the state resulting from these decisions (the current state). This implies that the optimal policy is computed “backwards”, in that we can split the optimal problem into 2 optimal control sub-problems, corresponding to 2 time intervals.
Figure 4: Trajectories for Minimal Time Case. Arrival time $T \approx 13$. 
Figure 5: Dynamic Programming Route
Example 6 (Shortest Path Problem) Consider the problem of travelling from A to B with the minimum total cost. The graph to be traversed, and the associated edge costs are given in the following diagram.

The approach taken is to work backwards from B – noting the minimum cost from there to B:

This results in the solution depicted in Figure 6.

For the solution of an optimal control problem, we split the overall time interval into two shorter time intervals. The solution of the first interval is dependent on the solution of the second interval, as follows:

Re-write the optimal control problem by splitting the integral at

\[
J(t_0) = \Phi(x(T), T) + \int_{t_0}^{t} L(x(\tau, u(\tau))d\tau + \int_{t}^{T} L(x(\tau, u(\tau))d\tau
\]
This gives the 2 sub-problems:

$$J_1(t_0) = V(x(t), t) + \int_{t_0}^{t} L(x(\tau, u(\tau)))d\tau$$

$$J_2(t) = \Phi(x(T), T) + \int_{t}^{T} L(x(\tau, u(\tau)))d\tau$$

where we define the function

$$V(x(t), t) = J_2(t)$$

Note that in the limit as $t \to T$, we find that $J_1 = J$ and $J_2 = \Phi(x(T), T)$. Alternatively, as $t \to t_0$, $J_1 = V$ and $J_2 = J$.

**Remark 6** The principle of optimality may now be restated as follows: let $u^*$ be the optimal control for the original control problem, for all $t \in [t_0, T]$ $u^*$ gives the optimal control for the control problems defined by $J_1$ and $J_2$.

**Definition 1** The function $V(x, t)$ is called the **value function** or the **cost to go**. It represents the value of the solution of the optimal control problem starting at $x$ at the time $t$. 
Consider the minimization of a value function $V(x, t)$ during $t, T$. Let $u^*$ be the optimal control for $T = t_f$. Then

$$V(x^*(t), t) = \Phi(x^*(t_f), t_f) + \int_t^{t_f} L(x^*(\tau), u^*(\tau))d\tau$$

$$= \min_u \{\Phi(x(t_f), t_f) - \int_t^{t_f} L(x(\tau), u(\tau))\}$$

Additionally, we have to fulfill

$$\dot{x} = f(x(t), u(t))$$

and the boundary condition

$$V(x, T) = \Phi(x(t_f), t_f)$$

It follows that

$$\frac{dV}{dt} = \min_u -L(x(t), u(t))$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \frac{dx}{dt} = \min_u -L(x(t), u(t))$$

$$\frac{\partial V}{\partial t} = \min_u \{-\frac{\partial V}{\partial x} f(x(t), u(t)) - L(x(t), u(t))\}$$

$$= -\min_u H(x, \frac{\partial V}{\partial x}, u)$$

**Definition 2** This equation is known as the Hamilton-Jacobi equation. This is a partial differential equation in $V$, with boundary condition $V(x, T) = \Phi(x(t_f), t_f)$. Solving this equation gives the solution of the optimal control problem with the control being a state feedback law given by

$$u^* = \arg \min H(x, \frac{\partial V}{\partial x}, u)$$

Note that the co-state $\lambda$ introduced in the previous section is then given by $\frac{\partial V}{\partial x}$.

When the Hamilton-Jacobi-Bellman equation is to be solved in practical cases, numerical methods seem appropriate. However, the solution of the Hamilton-Jacobi equation by numerical methods is only tractable for low dimensional systems. For higher dimensional systems, the number of data
points required increases with the power of the dimension of the system (space and time discretization). For example, for a nonlinear system of dimension $n$ where the value function is to be determined on a hyper-cube $[-a, a]$ with a spatial discretization $\epsilon$, a total of $\frac{2a^n}{\epsilon}$ points will have to be stored. For even modest requirements the amount of data becomes very large and not tractable.