Adaptive control

Consider the following scalar \((x \in \mathbb{R})\) system

\[
\dot{x} = -ax + bu.
\]  

(1)

The values \(a\) and \(b\) are unknown, however we assume that \(b > 0\), i.e. the sign of \(b\), is given. The objective is to make the system track the following known reference model

\[
\dot{x}_m = -a_m x_m + b_m u_r
\]

where \(a_m > 0\) and the reference input \(u_r\) is known. Let us consider the controller to be given as

\[
u = \phi_1 u_r - \phi_2 x\]

(2)

where the parameters \(\phi_1\) and \(\phi_2\) are the ones that we would like to adjust adaptively online. Define the difference between the system and the reference model as

\[
e = x - x_m\]

(3)

**Problem 1 : MIT rule design**

- Substitute the value of \(u\) in (2) into equation (1), and derive the transfer function from \(x\) to \(u_r\) (under zero initial condition).
- Using the transfer function derived above and the definition of \(e\) in (3), derive the following quantities: \(\frac{\partial E}{\partial \theta_1}\) and \(\frac{\partial E}{\partial \theta_2}\). Then, use these quantities to form the adaptation equations obtained via the MIT rule, i.e. minimize \(J(\theta) = \frac{1}{2} e^2\). Can you implement the MIT controller that you derived? Why?
- Use the following nominal approximation \(a + b\theta_2^2 \approx a_m\) in your answers for the second part. Can you implement the resulting adaptation laws now? What is the final MIT rule controller?

**Solution :**

- The closed-loop plant equation is

\[
\dot{x} = -(a+b\theta^2) x + b\theta_1 u_r \implies X(s) = \frac{b\theta_1}{s+a+b\theta_2^2} U_r(s)
\]

- As such, we have that

\[
\frac{\partial E}{\partial \theta_1} = \frac{b}{s+a+b\theta_2^2} U_r, \quad \frac{\partial E}{\partial \theta_2} = -\frac{b^2 \theta_1}{(s+a+b\theta_2)^2} U_r = -\frac{b}{s+a+b\theta_2} X
\]

Since we do not know \(a\) and \(b\), the partial derivatives above cannot be used in the MIT-based adaptation given by

\[
\dot{\theta} = -\gamma E \frac{\partial c}{\partial \theta}.
\]

- Using the approximation \(a + b\theta_2 \approx a_m\), we obtain the following adaptation rules instead

\[
s\theta_1 = -\gamma' \left( \frac{a_m}{s+a_m u_r} \right) E
\]

\[
s\theta_2 = -\gamma' \left( -\frac{a_m}{s+a_m x} \right) E
\]
where $\gamma' = \gamma \frac{b}{a_m}$. The sign of $b$ is indeed crucial for this scheme to work!

**Problem 2 : Lyapunov-based design**

Consider the following Lyapunov function candidate

$$V(e, \phi_1, \phi_1) = \frac{1}{2} \left( e^2 + \frac{1}{b\gamma} (b\theta_2 + a - a_m)^2 + \frac{1}{b\gamma} (b\theta_1 - b_m)^2 \right).$$

Design the update equations for $\theta_1$ and $\theta_2$ so that we obtain a stable system, i.e. $\dot{V} \leq 0$. *Hint : Use the following equation*

$$\dot{e} = -a_m e - (b\theta_2 + a - a_m) x + (b\theta_1 - b_m) u_r.$$

**Solution :**

Take the derivative of $V$ and substitute the given expression for $\dot{e}$ to obtain

$$\dot{V} = -a_m e^2 + \frac{1}{\gamma} (b\theta_2 + a - a_m) (\dot{\theta}_2 - \gamma xe) + \frac{1}{\gamma} (b\theta_1 - b_m) (\dot{\theta}_1 + \gamma u_r e).$$

Then pick the adaptation parameters in order to obtain $\dot{V} \leq 0$, i.e.

$$\dot{\theta}_1 = -\gamma u_r e$$
$$\dot{\theta}_2 = \gamma xe.$$

This guarantees that $(e, \theta_1, \theta_2)$ is bounded. With some work, one can show that $e \to 0$ as $t \to \infty$, but not $\theta_1$ and $\theta_2$. 