17 Sliding mode control

Consider the problem of doing setpoint control for a system of the form

\[ x^{(n)} = f(x) + b(x)u \]

That is equivalent to

\[
\begin{pmatrix}
\dot{x}_1 \\
\vdots \\
\dot{x}_{n-1} \\
\dot{x}_n
\end{pmatrix} =
\begin{pmatrix}
x_2 \\
\vdots \\
x_n
\end{pmatrix}
+ f(x) + b(x)u,
\text{ with } x = \begin{pmatrix}
\dot{x}_1 \\
\vdots \\
\dot{x}_{n-1} \\
\dot{x}_n
\end{pmatrix}
\]

The system \( \Sigma \) is a diagonal nonlinear system.

Consider that \( f(x) \) and \( b(x) \) are uncertain. Problem is to determine a control that solves the setpoint tracking problem and is also robust with respect to uncertainties in \( f(x) \) and \( b(x) \).

Define \( x_d \) (the setpoint) and \( \bar{x} \) (the error signal) the difference between \( x \) and \( x_d \):

\[
x_d = (x_d, \dot{x}_d, \ddot{x}_d, \ldots)
\]

\[
\bar{x} = x - x_d
\]

If we can restrict the dynamics of the system to lie on a well behaved surface, then the control problem is greatly simplified. The surface is called the sliding mode, and is defined in such a way that the error dynamics are exponentially stable when the system is restricted to lie on this surface.

The control problem then reduces the problem of driving the system to this surface, and then ensuring that it stays on this surface all the time.

Define the sliding mode \( S(t) \) as follows:

\[
S(t) = \{ x | s(x, t) = 0 \}
\]

where \( s(x, t) \) is defined by

\[
s(x, t) = \left( \frac{d}{dt} + \lambda \right)^{n-1} \bar{x}(t), \quad \lambda > 0
\]
Note that on the surface $S(t)$, the error dynamics are governed by the equation

$$\left(\frac{d}{dt} + \lambda\right)^{n-1} \ddot{x}(t) = 0$$

On this surface, the error will converge to 0 exponentially.

This implies that if there exists a control input $u(t)$ such that $x(t)$ is in $S(t)$ it follows that $x(T)$ is in $S(T)$ for all $T > t$ and the error will converge exponentially to 0 for this control input.

**Remark 12.1:** The choice of $s(x,t)$ is somewhat arbitrary. You may choose any error dynamic which leads to exponentially stable behavior

$$s(x,t) = p\left(\frac{d}{dt}\right) \ddot{x}$$

Where $p(s)$ is a polynomial with all zeros in the left half plane, then the error dynamics will converge exponentially. For example, we could have

$$p(s) = s^2 + s + 1$$

$$p\left(\frac{d}{dt}\right) \ddot{x} = \dddot{x} + \ddot{x} + 1$$

The strategy to converge to the sliding mode is that we add something to $u(t)$, which will drive us to the sliding mode in finite time.

In summary, the motion consists of a reaching phase during which trajectories starting off the manifold $s = 0$ move toward it and reach it in finite time, followed by a sliding phase during which the motion is confined to the manifold $s = 0$ and the dynamics of the system are represented by the reduced-order model. The manifold $s = 0$ is called the sliding manifold and the control law $u = -\beta(x) \text{sgn}(s)$ is called sliding control mode.

**Example 12.1:** Consider the second order system

$$\ddot{x} = f(\dot{x}, x) + u$$

where $x$ is a scalar. Let $f(\dot{x}, x)$ be an uncertain function, where only and estimate of the true state equation $\hat{f}(\dot{x}, x)$ is known, and that the error may be bounded

$$\left|f(\dot{x}, x) - \hat{f}(\dot{x}, x)\right| \leq F(\dot{x}, x), \forall \dot{x}, x$$
Define the sliding mode:

\[ s = \dot{x} + \lambda \ddot{x} = \left( \frac{d}{dt} + \lambda \right)^{n-1} x, \quad n = 2 \]

then

\[ \dot{s} = \dot{x} - \dot{x}_d + \lambda \ddot{x} = f(\dot{x}, x) + u - \dot{x}_d + \lambda \ddot{x} \]

Choose

\[ \hat{u} = -\hat{f}(x, \dot{x}) + \dot{x}_d - \lambda \ddot{x}, \quad f = \hat{f} \quad \text{and} \quad \dot{s} = 0 \]

and define the control

\[ u = \hat{u} - k \text{sgn}(s) \]

We consider \( s(x,t) \) to be a measure of how far we are from the sliding mode. In order to force the system to stay on the sliding mode, we choose \( u(t) \) such that \( \dot{s} = 0 \).

To prove convergence to the sliding mode, consider the derivative of the distance of the point from the sliding mode \( \frac{1}{2} s^2 \).

\[ \frac{1}{2} \frac{d}{dt} s^2 = \dot{s} \cdot s \]

\[ = \left( f(\dot{x}, x) - \hat{f}(\dot{x}, x) - k \text{sgn}(s) \right) s \]

\[ = \left( f(\dot{x}, x) - \hat{f}(\dot{x}, x) \right) s - k |s| \]

and for \( k(x, \dot{x}) = F(x, \dot{x}) + \eta \) we have

\[ \frac{1}{2} \frac{d}{dt} s^2 \leq -\eta |s| \]

With the control \( u = \hat{u} - k \text{sgn}(s) \) we achieve convergence to the sliding mode. In order to cope with uncertainty, we choose a higher gain \( k \) in the control.
**Example 12.2:** Consider the second order system

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= h(x) + g(x)u
\end{align*}
\]

where \( h \) and \( g \) are unknown nonlinear functions and \( g(x) \geq g_0 > 0 \) for all \( x \). We want to design a state feedback control law to stabilize the origin. Suppose we can design a control law that constrains the motion of the system to the manifold (or surface) \( s = a_1 x_1 + x_2 = 0 \). On this manifold, the motion is governed by \( \dot{x}_1 = -a_1 x_1 \). Choosing \( a_1 > 0 \) guarantees that \( x(t) \) tends to zero as \( t \) tends to infinity and the rate of convergence can be controlled by choice of \( a_1 \). The motion on the manifold \( s = 0 \) is independent of \( h \) and \( g \).

The variable \( s \) satisfies the equation

\[
\dot{s} = a_1 \dot{x}_1 + \dot{x}_2 = a_1 x_2 + h(x) + g(x)u
\]

Suppose \( h \) and \( g \) satisfy the inequality

\[
\left| \frac{a_1 x_2 + h(x)}{g(x)} \right| \leq \rho(x), \quad \forall x \in \mathbb{R}^2
\]

for some known function \( \rho(x) \). With \( V = (1/2)s^2 \) as a Lyapunov function candidate for \( \dot{s} = a_1 x_2 + h(x) + g(x)u \), we have

\[
\dot{V} = s \dot{s} = s \left[ a_1 x_2 + h(x) \right] + g(x) su \leq g(x)|s|\rho(x) + g(x)su
\]

Taking

\[
u = -\beta(x) \text{sgn}(s)
\]

where \( \beta(x) \geq \rho(x) + \beta_0, \beta_0 > 0 \), and

\[
\text{sgn}(s) = \begin{cases} 
1, & s > 0 \\
0, & s = 0 \\
-1, & s < 0
\end{cases}
\]

yields

\[
\dot{V} \leq g(x)|s|\rho(x) - g(x)\left[ \rho(x) + \beta_0 \right] s \text{sgn}(s) = -g(x)\beta_0 |s| \leq -g_0 \beta_0 |s|
\]

Therefore, the trajectory reaches the manifold \( s = 0 \) in finite time, and once on the manifold, it cannot leave it, as seen from the inequality \( \dot{V} \leq -g_0 \beta_0 |s| \).
The next figure represents a typical phase portrait under sliding mode control

![Phase Portrait](image)

We have to note that the controller is discontinuous at $s = 0$. Due to the effects of sampling, switching and delays in the devices used to implement the controller, respectively in the simulation engines used when modelling the controlled system, sliding mode control suffers from chattering.

The next figure shows how delays can cause chattering. It depicts a trajectory in the region $s > 0$ heading toward the sliding manifold $s = 0$. It first hits the manifold at a point $a$. In ideal sliding mode control, the trajectory should start sliding on the manifold from a point $a$. In reality, there will be a delay between the time the sign of $s$ changes and the time the control switches. During this delay period, the trajectory crosses the manifold into the region $s < 0$.

![Chattering](image)

Chattering results in low control accuracy, high heat losses in electrical power circuits and high wear of moving mechanical parts. It may also excite unmodeled high frequency dynamics, which degrades the performance of the system and may even lead to instability.
There are many strategies used to avoid chattering, \textit{e.g.} you can introduce a boundary layer. Here, the $\text{sgn}$ function is made continuous by using a piecewise linear approximation

Within the boundary layer you have exponentially convergence to the sliding mode. You rely on continuity arguments to show that the system will still converge.