A Fast Method for Real-Time Chance-Constrained Decision with Application to Power Systems

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***Abstract*—In this paper, we consider chance-constrained decision problems with a specific structure: on one hand, we assume that some prior information about the unknown parameters of the decision problem is known, in the form of samples; on the other hand, we assume that it is possible to gather further information regarding the true value of these parameters via measurements. We specialize the scenario approach so that the apriori samples can be efficiently used, together with the available measurement, to generate the feasible region where chance constraints are satisfied. This results in a two-phase algorithm, composed of an offline pre-processing of the samples, followed by an online part that needs to be performed as soon as the measurement is available. This online part is computationally extremely lightweight, both in terms of computation time and of memory footprint, and is therefore suited for implementation in embedded systems. As an application of choice, we consider the control of microgenerators in a power distribution grid.**

***Index Terms*—Stochastic optimization, computational methods, uncertain systems, scenario approach, power systems.**

I. INTRODUCTION

MANY engineering problems can be reformulated as instances of *decision under uncertainty*, that is, the problem of taking a cost-effective decision that satisfies some given constraints, when the problem parameters are uncertain.

One natural approach consists in taking a decision which is guaranteed to be feasible for any possible admissible value of the unknown parameters, therefore adopting a worst-case paradigm. In some applications, this *robust* approach yields tractable programs, for which an optimal solution can be found. Such a solution can be very conservative in terms of achievable performance: the decision may be affected by extreme values of the parameters that are very unlikely but have severe effects on the feasible region in which the optimal decision is sought for.

A second approach consists in formulating what is called a *chance-constrained decision problem*. In these problems, it is tolerated that the constraints of the problem can be violated for a set of parameter values that has minimal probabilistic measure in the parameter space, and is therefore very unlikely to realize. This approach allows to trade off *security* (intended as the probability of violating the constraints) for *performance* (the cost of the decision). Chance-constrained problems are in general non-convex and hard to solve, even if the original problem with known parameter values is convex.

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However, they can be effectively tackled by adopting the so-called *scenario approach*, in which the stochastic constraints are replaced by deterministic constraints, obtained by sampling the parameter uncertainty. If the number of constraints is sufficiently large, feasibility in the chance-constrained sense can be guaranteed with high confidence [1]–[3]. Conversely, it is also possible to derive guarantees on the cost of the solution obtained via this approach [4].

In this paper, we consider a variation of the scenario approach which arises when (i) the chance-constrained decision problem needs to be solved in a very limited amount of time or even online (ii) new information on the uncertain parameters can be obtained at the time of decision. This is the case, for example, when the chance-constrained decision is used to actuate a plant subject to disturbances, and on which some measurements are available in real time. In such an online setup, the scenario approach is not directly applicable, as samples from the conditional distribution of the unknown parameters are not readily available. We show how the scenario approach can be modified into a two-phase algorithm: in an offline phase, the samples of the uncertain parameters are pre-processed and encoded in an efficient polytopic representation; the online phase, to be executed when the measurement becomes available, is computationally extremely lightweight, and returns an approximation of the exact feasibility region in the decision space. The approximation is shown to be exact in the case of Gaussian distribution of the parameters. The effectiveness of the proposed method in the case of non-Gaussian distributions is illustrated in simulations.

As an application of choice, we consider the real-time operation of power distribution grids, and in particular the problem of curtailment of renewable generation. The expensive nature of these decisions (e.g., curtailing carbon-free energy) motivates the need of precise assessment of risk, in order to avoid unnecessarily conservative decisions. We show that it is possible to use, in a systematic way, both historical data and real-time measurements, in order to take cost-effective decisions with security guarantees. Related examples of chance-constrained decision in power system can be found for example in [5]–[9]. Preliminary work in this direction has been presented in [10], where however no measurements were considered.

The paper is organized as follows. In Section II we briefly review the scenario approach, and we formulate the chance-constrained decision problem with measurements. Our main contribution is presented in Section III, where we show how the posterior distribution of the unknown parameters can be approximated, and we propose a fast algorithm for the solution of the chance-constrained decision problem. In Section IV we illustrate the effectiveness of the proposed algorithm for real-time operation of power distribution grids.

A. Mathematical preliminaries

We briefly review some definitions that will be useful for the technical derivation of the proposed strategy.

A set $\mathcal{P} \subset \mathbb{R}^n$ is called a *polyhedron* if it is the intersection of m closed half-spaces

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax \leq b\}, \qquad A \in \mathbb{R}^{m\times n},\ b \in \mathbb{R}^m. \tag{1}$$

Here and in the rest of the paper, the $\leq$ relation is to be intended element-wise. A bounded polyhedron is called a *polytope*. We say that the representation (1) for $\mathcal{P}$ is *redundant* if there exists a smaller set of $m' < m$ closed half-spaces whose intersection is equal to $\mathcal{P}$. We call it *minimal* otherwise. Finally, we recall that the intersection of a finite number of polytopes is a polytope.

We refer the interested reader to [11], [12] for a presentation of the efficient numerical algorithms that allow manipulating polytopes, and in particular to compute minimal representations and intersections.

II. Problem statement

A. The scenario approach for chance-constrained decision

We consider the chance-constrained decision problem

$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{subject to} && \mathbb{P}\left[Ax + Bw \leq z\right] \geq 1 - \epsilon \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^n$ is the decision variable, $f(x)$ is a convex cost, $w \in \Omega \subseteq \mathbb{R}^m$ is an unknown disturbance modeled as a random variable, $z \in \mathbb{R}^l$ is a constant term. We assume that the support Ω of the random variable w is endowed with a σ-algebra $\mathcal{D}$ and that $\mathbb{P}$ is defined over $\mathcal{D}$. Finally, $\epsilon \in (0, 1)$ is the desired constraint violation probability.

General chance-constrained decision problems are nonconvex and often computationally intractable. Notice that we have assumed linear constraints affine in the random variable. In this case and whenever the underlying distribution of w is known, analytical results are available providing conditions for the chance-constrained problem to be reformulated as a convex problem [13, Ch. 8.3] [14]. In any other case, the scenario approach is an effective tool to convert stochastic programs of this kind into deterministic problems of the form

$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{subject to} && Ax + Bw^{(i)} \leq z \qquad i = 1, \ldots, N \end{aligned} \tag{3}$$

where $\{w^{(i)}\}$ are N samples of the stochastic disturbance. If N is large enough, then (3) is equivalent to (2) in the sense of the following result, that descends directly from [3, Theorem 1] (an extension of [2, Theorem 1]).

Theorem 1. *Let us define the positive constants*

ϵ *violation probability* $\in (0, 1)$
β *confidence level* $\in (0, 1)$

and consider the optimization problems (3)*. Let N satisfy*

$$\sum_{i=0}^{n-1} \binom{N}{i} \epsilon^i (1-\epsilon)^{N-i} \leq \beta,$$

where n is the dimension of the decision variable x. If the solution x^* to (3) *exists, then with probability larger than $1 - \beta$, it satisfies*

$$\mathbb{P}\left[Ax^* + Bw \leq z\right] \geq 1 - \epsilon.$$

The optimization problem (3) inherits the convexity of the individual deterministic constraints $Ax + Bw^{(i)} \leq z$, $i = 1, \ldots, N$. These half-spaces define a polyhedron with a typically very large number of redundant constraints. In the rest of the paper we assume that this polyhedron is bounded (and therefore a polytope) and non-empty, so that the solution x^* exists and Theorem 1 applies.

The scenario approach is remarkably distribution-free, meaning that no assumptions are made on the probability distribution of the disturbance w. The information about the distribution of w is still implicitly present, via the quantities $\{w^{(i)}\}$, that need to be sampled according to such distribution. This feature of the scenario approach makes it very attractive for those applications in which a reliable first-principle model of the disturbance is not available, but historical data can be used instead.

B. Scenario approach with measurements

In certain applications, online information about the disturbance w may be available. For example, although a prior information on the distribution of w may be available beforehand, some direct measurement may be possible at the time of the decision. We formalize the problem of chance constrained decision with measurements as

$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{subject to} && \mathbb{P}\left[Ax + Bw \leq z \mid Hw = y\right] \geq 1 - \epsilon, \end{aligned} \tag{4}$$

where $y = Hw$ is a linear measurement of the disturbance, in which H is full row-rank, and $\mathbb{P}[\,\cdot\,|\,\cdot\,]$ means conditional probability. A straightforward application of the scenario approach, as in (3), would yield a deterministic optimization program of the form

$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{subject to} && Ax + Bw_y^{(i)} \leq z \qquad i = 1, \ldots, N \end{aligned} \tag{5}$$

where $w_y^{(i)}$ are samples of the *conditional probability distribution* determined by the measurement $y = Hw$.

This last setup seems to nullify the effectiveness of the scenario approach for real-time operations, as the samples $\{w_y^{(i)}\}$ need to be generated only after the measurement y is available. The use of historical samples makes the integration of this kind of new information difficult. Moreover, the resulting optimization problem (5) still presents a large number of typically redundant constraints, which poses a computational challenge to the direct use of the scenario approach for fast real-time decisions.

In the next section, we will show how both these issues can be successfully resolved by means of an offline pre-processing phase of the samples, followed by an online measurement-driven decision step.

III. A FAST METHOD FOR CHANCE CONSTRAINED DECISION

A. Approximate conditioning via affine transformation

Let μ and Σ be the mean and covariance, respectively, of the random variable w, i.e.,

$$\mu = \mathbb{E}[w], \qquad \Sigma = \mathrm{cov}[w].$$

Consider the following affine transformation of the random variable w, induced by the measurement $y = Hw$:

$$\hat{w}_y = w + K(y - Hw) \tag{6}$$

where

$$K = \Sigma H^T \left(H\Sigma H^T\right)^{-1}.$$

Remark. *We assume H to be full row-rank in order to ensure non-singularity of the term $H\Sigma H^T$. If the linear measurement is noisy, $y = Hw + \eta$, then the assumption on H can be relaxed, as the constraint in* (4) *becomes*

$$\mathbb{P}\left[Ax + \begin{bmatrix}B & 0\end{bmatrix}\begin{bmatrix}w\\ \eta\end{bmatrix} \leq z \mid \begin{bmatrix}H & I\end{bmatrix}\begin{bmatrix}w\\ \eta\end{bmatrix} = y\right] \geq 1 - \epsilon.$$

We have the following result, in the case of Gaussian w.

Proposition 1. *Let w be a normally distributed random variable, and let $\hat{w}_y$ be defined as in* (6). *Then the distribution of $\hat{w}_y$ is equal to the conditional distribution of w given the measurement $y = Hw$.*

Proof. Let us first consider two jointly Gaussian random variables w and y, with moments

$$\tilde{\mu} = \begin{bmatrix}\mu_w\\ \mu_y\end{bmatrix}, \qquad \tilde{\Sigma} = \begin{bmatrix}\Sigma_{ww} & \Sigma_{wy}\\ \Sigma_{yw} & \Sigma_{yy}\end{bmatrix}.$$

The conditional probability distribution $p(w|y)$ is Gaussian with moments (see [15])

$$\Sigma_{w|y} = \Sigma_{ww} - \Sigma_{wy}\Sigma_{yy}^{-1}\Sigma_{yw} \tag{7}$$

$$\mu_{w|y} = \mu_w - \Sigma_{wy}\Sigma_{yy}^{-1}(y - \mu_y). \tag{8}$$

To prove the statement of the proposition, we evaluate (7) and (8) for the deterministic relation $y = Hw$, and therefore $\Sigma_{ww} = \Sigma$, $\Sigma_{wy} = \Sigma H^T = \Sigma_{yw}^T$, $\Sigma_{yy} = H\Sigma H^T$, $\mu_w = \mu$. Simple substitution yields

$$\Sigma_{w|y} = \Sigma - KH\Sigma$$
$$\mu_{w|y} = \mu + K(y - H\mu).$$

These are the same moments as variable $\hat{w}_y$. As $\hat{w}_y$ is also Gaussian (affine transformation of a Gaussian variable), the two random variables have the same distribution. □

The interpretation of Proposition 1 is given in Figure 1, where the transformation (6) is applied to samples of the distribution w. The transformed samples of a Gaussian distribution w provide an exact sampling of the conditional distribution. On the other hand, transformed samples of non-Gaussian distribution (such as the uniform distribution in the right panel) can be considered as an approximate sampling of the conditional distribution.

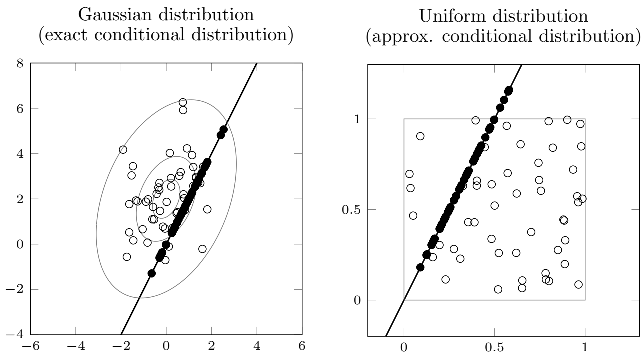


Fig. 1. The figure compares the transformation (6) applied on the left to normally distributed samples and on the right to uniformly distributed samples (the empty circles are samples of w). The thin lines are level curves of the probability distribution of w. The black circles represent the outcome of the transformation and are therefore samples of the distribution $\hat{w}_y$. The thick line is the subspace defined by the measurement $y = Hw$.

The quality of this approximation is illustrated in Figure 2 for two examples of non-Gaussian prior distributions. In the first case, we obtained a bimodal distribution by applying a discrete random offset to a Gaussian distribution. By applying transformation (6), we obtain an approximate posterior distribution that features the same bimodal nature as the true posterior distribution. In the second case, we considered a uniform distribution on an annulus. The resulting approximate posterior distribution features a compact support (as the true posterior distribution), although the approximation error is larger in this case. Both the multi-modality of a distribution and its compact support would have been lost if we followed a conventional and often-used engineering approach [15], namely to approximate (or assume) the prior distribution by a Gaussian one.

B. Affine transformation of the feasible region

The major computational complexity of the scenario approach (5) lies in the computation of the feasible polytope

$$\mathcal{P}_y = \left\{x \in \mathbb{R}^n \,\middle|\, Ax + Bw_y^{(i)} \leq z,\ i = 1, \ldots, N\right\}, \tag{9}$$

as $\{w_y^{(i)}\}$ are samples of the conditional distribution. Based on the findings that we just presented in Section III-A, we adopt $\hat{w}_y$ as an approximation of w_y (exact, in the case of Gaussian w). Each half-plane in (9) becomes, using (6),

$$Ax + B\left(w^{(i)} + K(y - Hw^{(i)})\right) \leq z, \quad i = 1, \ldots, N.$$

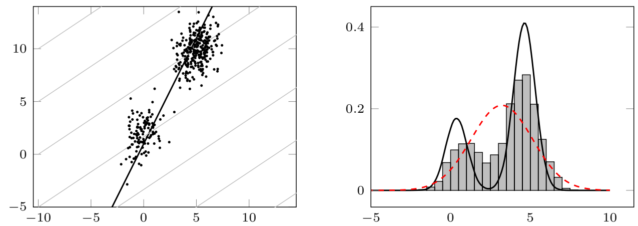
This allows us to define the following augmented polytope

$$\hat{\mathcal{P}} = \Big\{ \left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right] \in \mathbb{R}^{n+p} \,\Big|\, Ax + BKy + B(I - KH)w^{(i)} \leq z,\ i = 1, \ldots, N\Big\}, \tag{10}$$

where $\{w^{(i)}\}$ are samples of the unconditional distribution.

The polytope $\hat{\mathcal{P}}$ can be constructed *before* the measurement y is available and all redundant constraints can be eliminated

Bimodal distribution	Mean	Variance	Skewness	Kurtosis
True posterior	3.35	4.23	-0.74	2.00
Gaussian approximation	3.20	3.57	0	3
Affine transformation	3.20	3.57	-0.54	2.35



Annular distribution	Mean	Variance	Skewness	Kurtosis
True posterior	-0.6	32.9	0	1.08
Gaussian approximation	-0.6	17.8	0	3
Affine transformation	-0.6	17.8	0	1.60

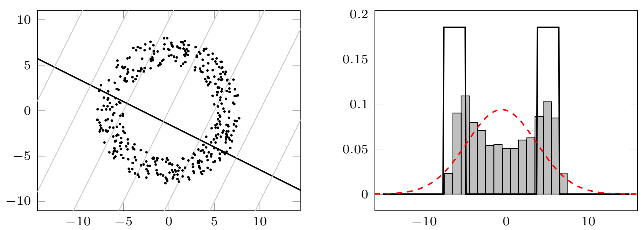


Fig. 2. The quality of the approximate conditioning proposed in Section III-A is assessed for two non-Gaussian random variables. For both distributions, in the left panel, we plotted some samples and the subspace spanned by the available measurement $y = Hw$ (thick line). The thin lines represent the direction of the projection (6). The right panel shows the true posterior distribution (solid black line), together with the sampling obtained by applying (6) to the original (apriori) samples. The dashed red line represents the posterior distribution that one would obtain by approximating the prior with a Gaussian distribution.

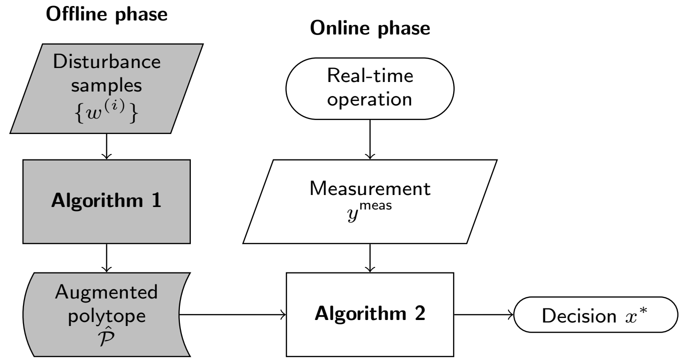


Fig. 3. Flowchart representing the proposed two-phase algorithm.

Algorithm 1: Construct augmented feasible polytope

Input
- Set of samples $w^{(i)}, i = 1, \dots, N$
- Measurement matrix H

Output
- Augmented feasible polytope $\hat{\mathcal{P}}$

Compute sample covariance matrix Σ ;
Compute $K = \Sigma H^T \left(H \Sigma H^T\right)^{-1}$;
for $i = 1, \dots, N$ **do**
Construct
$$\hat{\mathcal{P}}^{(i)} = \left\{ \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \,\middle|\, Ax + BKy + B(I - KH)w^{(i)} \leq z \right\}$$
end
Compute minimal representation of $\hat{\mathcal{P}} = \bigcap_i \mathcal{P}^{(i)}$;

Algorithm 2: Online chance constrained decision

Input
- Augmented feasible polytope $\hat{\mathcal{P}}$
- Measurement y^{meas}

Output
- Solution x^* of the decision problem (4)

Slice $\hat{\mathcal{P}}$ at $y = y^{\text{meas}}$ to obtain $\hat{\mathcal{P}}_y = \left\{ x \middle| \left[\begin{smallmatrix} x \\ y^{\text{meas}} \end{smallmatrix}\right] \in \hat{\mathcal{P}} \right\}$;
$x^* \leftarrow \arg\min f(x)$ subject to $x \in \hat{\mathcal{P}}_y$;

in this offline phase. The dimension of the reduced object is the sum of the number of decision variables and the number of measurements, hence computationally tractable.

To obtain an approximation $\hat{\mathcal{P}}_y$ of the feasible polytope $\mathcal{P}_y$ for the decision variable x, as defined in (9), it suffices to *slice* $\hat{\mathcal{P}}$ at the measured value y^{meas}, as soon as it is available. This online operation corresponds to just adding the linear equalities $y = y^{\text{meas}}$ and eliminating the corresponding p coordinates, with minimal computational effort.

C. A two-phase decision algorithm

We propose an algorithm made of an *offline phase*, which can be performed before the measurement y becomes available, based on historical samples of the disturbance, and an *online phase*, which has to be performed in real-time, as soon as the measurement is available. This two-phase architecture is represented by the flowchart in Figure 3, where the central role of the augmented polytope $\hat{\mathcal{P}}$ is evident (being the piece of information that needs to be stored and retrieved).

The two core procedures of the offline and online phase of the decision process are described in Algorithm 1 and Algorithm 2, respectively.

It needs to be pointed out that computing a minimal representation of the feasible region in Algorithm 1 is computationally much more expensive than solving an optimization program subject to all the constraints, even including the redundant ones [16]. Removing redundant constraints has, however, the added benefit of greatly reducing the size of the representation, with obvious benefits both for memory use and for communication requirements in case of embedded systems. Moreover, in some cases, a tractable representation of the feasible reasons is needed, rather than just the solution to the optimization program. Examples of such cases in power systems applications are discussed in [17, Section IV].

IV. Application: Power distribution grids

A. Active power curtailment for overvoltage prevention

The increasing presence of microgeneration poses unprecedented challenges to the operation of power distribution grids [18]. Distribution network operators (DNOs) are increasingly required to perform a real-time assessment of the state of the network, to make proactive decisions by controlling the available actuation devices, and to take remedial actions. Such a decision process cannot be effectively tackled by specializing the traditional tools available for deterministic Optimal Power Flow programming, primarily because the DNO does not have access to the real-time state of the distribution grid, which is mostly unmonitored. Instead, this problem can be cast as a chance-constrained decision which directly encodes the uncertainty on the state.

We consider the problem of voltage violations caused by distributed generation. To prevent overvoltage, the DNO can decide to limit the power injection of some generators – an expensive decision, given the extremely low marginal production cost of renewable power sources. This decision needs to be updated repeatedly, based on the latest field measurements coming from the grid.

We introduce the notation:

$\mathcal{B}$ set of all grid buses
$\mathcal{G}$ subset of buses where a generator is connected
v_i voltage magnitude at bus $i \in \mathcal{B}$
p_i^d power demand at bus $i \in \mathcal{B}$
q_i^d reactive power demand at bus $i \in \mathcal{B}$
p_i^g power generation at bus $i \in \mathcal{G}$.

To model the relationship between the grid voltages and the power injections/demands, we linearize the power flow equations around the flat voltage profile ($v_i = 1, \forall i \in \mathcal{B}$), corresponding to the *Linear Coupled power flow model* (we refer to [19, Section V] for its derivation). Based on this approximation, voltage magnitudes can be expressed as

$$v = \mathbf{1} + Rp + Xq, \tag{11}$$

where R and X are the bus resistance and reactance matrices and where elements of the active power injection vector p are defined as $p_i = p_i^g - p_i^d$ (the terms p_i^g being present only for buses in $\mathcal{G}$). Similarly, $q_i = -q_i^d$.

The DNO wants to maximize the generated power from renewable sources, and therefore aims at solving the following chance-constrained decision problem.

$$\begin{aligned} \max \quad & \textstyle\sum_{i\in\mathcal{G}} p_i^g \\ \text{subject to} \quad & \mathbb{P}\left[v \le v^{\max} \text{ and } v \ge v^{\min}\right] \ge 1-\epsilon \\ & (11). \end{aligned}$$

This problem has the form of (2), and therefore we can apply the solution proposed in Section III.

B. Numerical simulations

As a test case, we adopted the one proposed in [19], consisting of the three-phase backbone of the standard IEEE 123-bus distribution test feeder (see Figure 4).

To model the uncertainty of the power demands, we consider the dataset available in the DiSC simulation framework [20], which has been obtained as anonymized data from the Danish DSO NRGi. It represents the power consumption of about 1200 individual households from the Danish city Horsens.

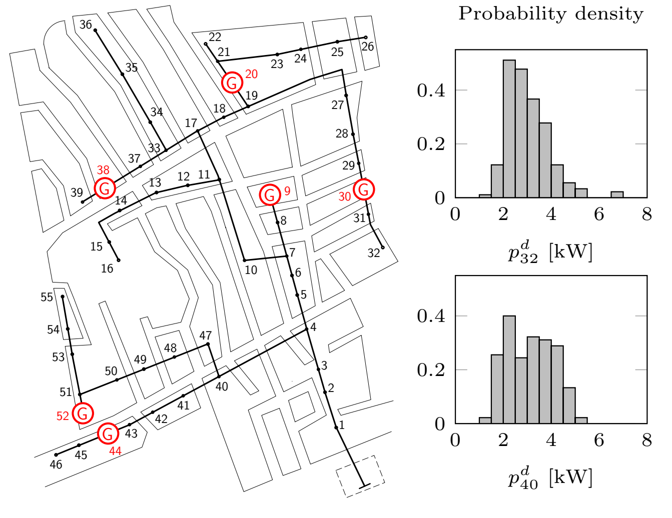


Fig. 4. The three-phase backbone of the IEEE 123-bus test feeder, with the distributed microgenerators added for this example. The right plots represent some typical distributions of the power demand of small aggregates of consumers (same time of the day, for 180 days).

The power demand at each bus is obtained by aggregating the power demands of 20 different households on 2011-06-29 at 15:00:00. We assume that the DNO has access to historical power demands of the same households in the previous six months, and can use those past measurements as samples for a scenario-based approach. Some examples of the distribution of these demands are reported in Figure 4. Interestingly, it can be seen that the typical demand is not exactly normally distributed: it is not symmetric, it shows longer tails for large power demands, and it has compact support (in particular, the demand is always positive).

The DNO has to guarantee that bus voltages across the entire grid remain between $v^{\max} = 0.95$ p.u. and $v^{\max} = 1.05$ p.u., with probability larger than $1 - \epsilon = 95\%$. It also has access to one scalar measurement, the total power demand, obtained at the point of coupling to utility grid:

$$y = \textstyle\sum_{i\in\mathcal{B}} p_i^d.$$

To gain some insight on the problem, we first consider the case in which only two generators are present, namely $\mathcal{G} = \{30, 38\}$, as this allows graphical representation of the results. In Figure 5 we plot the feasible region for the decision variable $x = \begin{bmatrix} p_{30}^g & p_{38}^g \end{bmatrix}^T$, in the case in which the measurement y is not used, and in the case in which y is used and takes the values 0 MW (no load) and 3 MW (typical load). Some comments are due.

- The feasible region without measurements is smaller than any other feasible region based on measurements. This is consistent with the intuition that more uncertainty implies more conservativeness in the decision.
- The case with $y = 0$ MW, corresponding to the unusual condition of zero load, results in a feasible region which does not include the solution of the chance-constrained problem without measurements. In fact, this example illustrates that violation is possible, with low probability, when a chance-constrained approach is adopted. It also

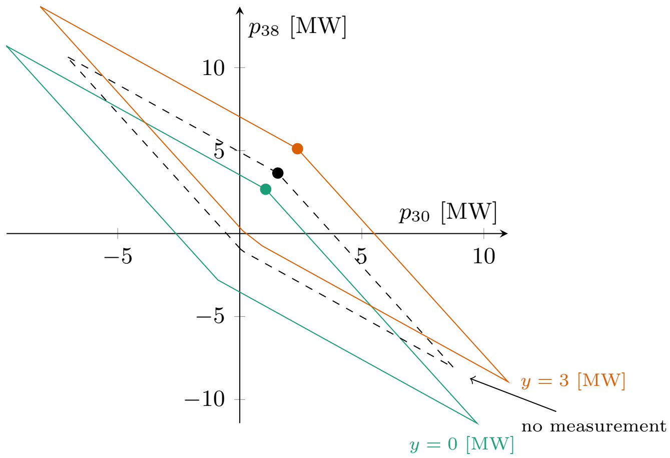


Fig. 5. Representation of the feasible polytopes and of the solutions (dots) of the chance-constrained problem solved via the proposed method, for different values of the measurement y, and without measurement.

Computation time (dual-core, 2.6 GHz processor, 64-bit OS)		
Offline	Construct augmented polytope $\hat{\mathcal{P}}$ Compute minimal representation of $\hat{\mathcal{P}}$	
	Total offline computation time	*55 min*
Online	Slice $\hat{\mathcal{P}}$ at $y = y^{\text{meas}}$ to obtain $\hat{\mathcal{P}}_y$ Solve LP defined on $\hat{\mathcal{P}}_y$	
	Total online computation time	*1.8 ms*
	Memory footprint	
Offline	Augmented polytope $\hat{\mathcal{P}}$	48620 constraints
Online	Minimal representation of $\hat{\mathcal{P}}$	12 constraints

TABLE I
COMPUTATIONAL COMPLEXITY OF THE PROPOSED ALGORITHM

shows how the incorporation of real-time measurements can be useful for security.

- For increasing values of the total demand y, larger amounts of generation can be tolerated by the grid. In fact, already when $y = 3$ MW (a typical demand for this grid), using this information in the chance-constrained decision allows achieving better performance, i.e., inject more power from the renewable sources.

We finally assess the computational complexity of the proposed method in the case of six generators, i.e., $\mathcal{G} = \{9, 20, 30, 38, 44, 52\}$. As the dimension of the decision variable is $n = 6$, we obtain from Theorem 1 that $N = 442$ samples of the power demands are needed. The resulting computation times and memory footprint of the proposed two-phase algorithm are reported in Table I. The complexity of the offline phase is significant. However, no stringent time and memory constraints should be expected there, as this phase can be completed on the historical samples. The online phase, on the other hand, requires minimal computational resources and can be easily executed by a microcontroller.

V. CONCLUSIONS

We considered the possibility of using the scenario approach to solve real-time chance-constrained decision problems in which new information on the unknown parameters of the problem becomes available via measurements. The affine nature of the constraints has been exploited to derive a variation of the scenario approach which does not require to re-sample the parameter space according to the conditional distribution. By pre-processing the samples, the chance constrained decision problem can be solved with extremely limited computational resources, making this approach attractive for large-scale systems with real-time control specifications. Finally, the proposed approach could be extended to setups in which control actions span a receding time horizon, as in Model Predictive Control of discrete-time systems.

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