On the steady-state behavior of low-inertia power systems ★

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Abstract: Whereas conventional power systems heavily rely on bulk generation by synchronous machines, future power systems will be comprised of distributed generation based on renewable sources interfaced by power electronics. A direct consequence of retiring synchronous generators is the loss of rotational inertia, which thus far was the dominant time constant in a power system, as well as the loss of the generator controls, which are the main source of actuation of the power grid. Prompted by these paradigm shifts, we study the dynamic behavior of a nonlinear and first-principle low-inertia power system model including detailed power converter models and their interactions with the power grid. In this paper, we focus particularly on the admissible steady-state behavior of such a low-inertia power grid and derive necessary and sufficient control specifications for power converters.

Keywords: power system dynamics, steady-state behavior, port-Hamiltonian systems.

1. INTRODUCTION

The electric power system is currently undergoing a major transition towards integration of large shares of distributed generation connected by power electronic converters. Today power systems operation is based on bulk generation by synchronous machines and heavily relies on their rotational inertia for robustness. In contrast, future power systems will be based on renewable sources, distributed generation, and power electronics.

A direct consequence of retiring synchronous generators is the loss of rotational inertia, which thus far was the main reason for the grid's stability and robustness to disturbances. This results in larger, and more frequent, frequency deviations and jeopardizes the stability of the power grid (Tielens and Hertem, 2016; Winter et al., 2015). At the same time, analysis of such phenomena is a challenging problem because the power system physics are highly nonlinear, large-scale, and contain dynamics on multiple time scales from mechanical and electrical domains. As a result, analysis and control of conventional power system are typically based on reduced order models of various degrees of fidelity (Sauer and Pai, 1998).

A widely accepted reduced model of conventional power systems is a structure-preserving multi-machine model, where each generator model is reduced to the swing equation describing the interaction between the generator rotor and the grid, which is itself modeled at quasi-steady-state via the nonlinear algebraic power balance equations. While this prototypical model has proved itself useful (Sauer and Pai, 1998) its validity for conventional power systems has always been a subject of debate; see (Caliskan and Tabuada, 2015; Venezian and Weiss, 2016) for recent discussions. Because the derivation of this model crucially relies on time-scale separations induced by the rotational inertia of the generators its validity for low-inertia power systems is highly questionable. For instance, if power electronics are modeled as a constant power source and all generators are removed only algebraic equations remain. Moreover, it is not clear which dynamics of power electronics devices need to be included, or how the AC signals which connect the power electronics to the grid enter into the swing equation.

One approach to mitigate the loss of rotational inertia is to use power electronic devices to provide virtual inertia. These control schemes typically measure AC signals which are not present in the swing equation (Zhong and Weiss, 2011; D'Arco and Suul, 2013; Sinha et al., 2015). As a first step towards overcoming these limitations, we propose a first-principles model of a power system containing synchronous machines, DC/AC inverters, as well as transmission line and voltage bus dynamics. The models of the generator and the network are based on the detailed port-Hamiltonian power system model proposed in Fiaz et al. (2013) and combined with an averaged DC/AC inverter model proposed by Jouini et al. (2016). In contrast to typical inverter models, we explicitly consider the dynamics of the DC-link capacitor, which is the dominant time constant of the DC/AC inverter dynamics.

We seek answers to similar questions as in our previous works on conventional power systems (Arghir et al., 2016; Groß et al., 2016): under what conditions does the system admit a steady-state in which all three-phase AC signals are balanced, sinusoidal, and of the same synchronous frequency. We provide an algebraic characterization that specifies the state variables, control inputs, and a synchronous frequency such that the dynamics of the power system coincide with the desired steady-state dynamics.

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This analysis constructively leads to necessary conditions that any controller for the power system has to satisfy in steady-state. Specifically, we show that the set of states and control inputs on which the dynamics coincide is control-invariant if and only if the DC current supplied to the inverter as well as all generator inputs are constant, and the inverter switching block operates at the synchronous frequency. Several heuristic inverter control strategies ("heuristic" in the sense that they are not constructed from a priori specifications), such as droop control (Dörfler et al., 2016), virtual oscillator control (Sinha et al., 2015), synchronverters (Zhong and Weiss, 2011), generator emulation (D'Arco and Suul, 2013), matching control (Jouini et al., 2016), and grid-following control (Tabesh and Iravani, 2009), implicitly satisfy these specifications in steady-state. Moreover, the steady-state specifications for the generators justify assumptions used in the stability analysis of multi-machine networks (Caliskan and Tabuada, 2014).

2. NOTATION AND PROBLEM SETUP

2.1 Notation

We use $\mathbb{R}$ and $\mathbb{N}$, to denote the set of real numbers and integers, and e.g. $\mathbb{R}_{>0}$ to denote the set of positive real numbers. For column vectors $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ we use $(x, y) = [x^\top \; y^\top]^\top \in \mathbb{R}^{n+m}$ to denote a stacked vector, and for vectors or matrices x, y we use $\operatorname{diag}(x, y) = \left[\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right]$. Furthermore, I_n denotes the identity matrix of dimension n, $\otimes$ denotes the Kronecker product, and $\|x\| = \sqrt{x^\top x}$ denotes the Euclidean norm. Matrices of zeros and ones of dimension $n \times m$ are denoted by $\mathbb{0}_{n\times m}$ and $\mathbb{1}_{n\times m}$, and $\mathbb{1}_n$ denotes a column vector of ones of length n.

2.2 Dynamical Model of a Power Network

The power system model used in this work consists of n_g generators with index set $\mathbb{G} = \{1, \dots, n_g\}$, n_i DC/AC inverters with index set $\mathbb{I} = \{n_g+1, \dots, n_g+n_i\}$, n_v voltage buses with index set $\mathbb{V} = \{1, \dots, n_v\}$, and n_t transmission lines with index set $\mathbb{T} = \{1, \dots, n_t\}$. The AC voltage buses are partitioned into generator buses $\mathbb{V}_g = \{1, \dots, n_g\}$, n_i inverter buses $\mathbb{V}_i = \{n_g+1, \dots, n_g+n_i\}$, and n_l load buses $\mathbb{V}_l = \{n_g+n_i+1, \dots, n_g+n_i+n_l\}$, i.e. $n_v = n_g+n_i+n_l$. The components of the power system as well as the main signals and parameters are depicted in Figure 1.

The model used in this manuscript combines a DC/AC inverter model proposed by Jouini et al. (2016) with a variant of the port-Hamiltonian power system model by Fiaz et al. (2013) used in Arghir et al. (2016). The reader is referred to these references for detailed derivations and discussions of the models. Based on the following assumption all three-phase AC signals are represented in (α, β) coordinates.

Assumption 1. All three-phase AC quantities are assumed to be balanced. Moreover, we assume that all three-phase electrical components (resistance, inductance, capacitance) have identical values for each phase.

Inverter Dynamics:
We consider a three-phase DC/AC inverter consisting of a DC-link capacitor, a switching block that modulates the DC-link capacitor voltage into an AC voltage, and an output filter. For the time scale of interest, we assume that the switching frequency is high enough and that the switching harmonics are suppressed by the output filter. By averaging the behavior of the switching block over one switching period, an averaged model of the inverter is obtained (Tabesh and Iravani, 2009; Jouini et al., 2016). The dynamics of the k-th inverter, with $k \in \mathbb{I}$, is given by:

$$\dot{q}_{I,k} = -G_{I,k} C_{I,k}^{-1} q_{I,k} + i_{sw,k}(\lambda_{I,k}, m_k) + i_{dc,k}, \tag{1a}$$
$$\dot{\lambda}_{I,k} = -R_{I,k} L_{I,k}^{-1} \lambda_{I,k} + C_k^{-1} q_k - v_{sw,k}(q_{I,k}, m_k), \tag{1b}$$

with DC-link capacitor charge $q_{I,k} \in \mathbb{R}_{\geq 0}$, output filter flux $\lambda_{I,k} = (\lambda_{I,\alpha,k}, \lambda_{I,\beta,k}) \in \mathbb{R}^2$. The AC-side capacitor, with charge $q_k = (q_{\alpha,k}, q_{\beta,k}) \in \mathbb{R}^2$, interconnects the inverter to the grid and will be described later in the model of the AC voltage bus dynamics. The control inputs of the inverter are the modulation signal $m_k \in \mathbb{R}^2$, which has to satisfy $\|m_k\| \leq 1$, and the current $i_{dc,k}$ supplied to the DC-link capacitor. In other words, we assume that $i_{dc,k}$ is supplied by a controllable source, e.g. a boost converter connected to photovoltaics and/or a battery.

The averaged output voltage $v_{sw,k}(q_{I,k}, m_k)$ and switching current $i_{sw,k}(\lambda_{I,k}, m_k)$ are given by:

$$i_{sw,k}(\lambda_{I,k}, m_k) = \tfrac{1}{2} \lambda_{I,k}^\top L_{I,k}^{-1} m_k, \tag{2a}$$
$$v_{sw,k}(q_{I,k}, m_k) = \tfrac{1}{2} C_{I,k}^{-1} q_{I,k} m_k \tag{2b}$$

The DC capacitance and conductance are denoted by $C_{I,k} \in \mathbb{R}_{>0}$ and $G_{I,k} \in \mathbb{R}_{>0}$ and $L_{i,k} = I_2 l_{I,k}$, $l_{I,k} \in \mathbb{R}_{>0}$ and $R_{I,k} = I_2 r_{I,k}$, $r_{I,k} \in \mathbb{R}_{>0}$ denote the inductance and resistance of the output filter.

Synchronous Generator Dynamics:
A generator with index $k \in \mathbb{G}$ is modeled by

$$\dot{\theta}_k = {M_k}^{-1} p_k \tag{3a}$$
$$\dot{p}_k = -D_k {M_k}^{-1} p_k - \tau_{e,k}(\lambda_k, \theta_k) + \tau_{m,k} \tag{3b}$$
$$\dot{\lambda}_k = -R_k \mathcal{L}_{\theta,k}^{-1} \lambda_k + \begin{bmatrix} C_k^{-1} q_k \\ v_{f,k} \end{bmatrix}, \tag{3c}$$

where $\lambda_k = (\lambda_{s,k}, \lambda_{f,k}) \in \mathbb{R}^3$ represents the stator flux linkage $\lambda_{s,k} = (\lambda_{\alpha,k}, \lambda_{\beta,k}) \in \mathbb{R}^2$ and rotor flux linkage $\lambda_{f,k} \in \mathbb{R}$, $p_k \in \mathbb{R}$ is the momentum of the rotor, and $\theta_k \in \mathbb{R}$ its angular displacement. The generator is actuated by the voltage $v_{f,k} \in \mathbb{R}$ across the excitation winding of the generator and the mechanical torque $\tau_{m,k} \in \mathbb{R}$ applied to the rotor. The electrical torque acting on the rotor is denoted by $\tau_{e,k}(\lambda_k, \theta_k) = \frac{\partial}{\partial \theta_k}(\frac{1}{2} \lambda_k^\top \mathcal{L}_{\theta,k}^{-1} \lambda_k)$. The inertia and damping of the rotor are given by $M_k \in \mathbb{R}_{>0}$ and $D_k \in \mathbb{R}_{>0}$, and the windings have resistance $R_k = \operatorname{diag}(R_{s,k}, r_{f,k})$ with $R_{s,k} = I_2 r_{s,k}$, $r_{s,k} \in \mathbb{R}_{>0}$, and $r_{f,k} \in \mathbb{R}_{>0}$. The inductance matrix $\mathcal{L}_{\theta,k} : \mathbb{R} \to \mathbb{R}^{3\times 3}$ is a function of the angle θ_k and defined based on the stator inductance $L_{s,k} = I_2 l_{s,k} \in \mathbb{R}^{2\times 2}_{>0}$, mutual inductance $L_{m,k} = (l_{m,k}, 0) \in \mathbb{R}^2_{>0}$, and rotor inductance $l_{r,k} \in \mathbb{R}_{>0}$:

$$\mathcal{L}_{\theta,k} = \begin{bmatrix} L_{s,k} & \mathcal{R}_{\theta_k} L_{m,k} \\ L_{m,k}^\top \mathcal{R}_{\theta_k}^\top & l_{r,k} \end{bmatrix}, \; \mathcal{R}_{\theta_k} = \begin{bmatrix} \cos(\theta_k) & -\sin(\theta_k) \\ \sin(\theta_k) & \cos(\theta_k) \end{bmatrix}.$$

For notational convenience, we introduce skew symmetric matrices $j \in \mathbb{R}^{2\times 2}$ and $\jmath \in \mathbb{R}^{3\times 3}$:

$$j = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \jmath = \begin{bmatrix} j & \mathbb{0}_{2\times 1} \\ \mathbb{0}_{1\times 2} & 0 \end{bmatrix}.$$

Based on this, the electrical torque $\tau_{e,k}(\lambda_k, \theta_k) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ can be expressed as follows:

$$\tau_{e,k}(\lambda_k, \theta_k) = \frac{1}{2}\lambda_k^\top (\mathcal{L}_{\theta,k}^{-1}\mathcal{J}^\top + \mathcal{J}\mathcal{L}_{\theta,k}^{-1})\lambda_k. \quad (4)$$

Moreover, the stator current $i_{s,k}$ and excitation current $i_{f,k}$ are given by $i_k = (i_{s,k}, i_{f,k}) = \mathcal{L}_{\theta,k}^{-1}\lambda_k \in \mathbb{R}^3$.

Interconnection Graph of the Transmission Network:
The AC voltage buses are interconnected by a transmission network. The topology of the transmission network is described by the (oriented) incidence matrix E of its associated graph (see e.g. Fiaz et al. (2013)). In the remainder, we consider the following partition of the incidence matrix $\mathcal{E} \in \{-1, 1, 0\}^{2n_v \times 2n_t}$:

$$\mathcal{E} = E \otimes I_2 = \begin{bmatrix} \mathcal{E}_{q,1} \\ \vdots \\ \mathcal{E}_{q,n_v} \end{bmatrix} = [\mathcal{E}_{T,1} \ \dots \ \mathcal{E}_{T,n_t}]. \quad (5)$$

AC Voltage Bus Dynamics:
The dynamics of the AC voltage bus connected to the generator with index $k \in \mathbb{V}_g$ are given by

$$\dot{q}_k = -G_k C_k^{-1} q_k - [I_2 \ \mathbb{0}_{2\times 1}]\,\mathcal{L}_{\theta,k}^{-1}\lambda_k - \mathcal{E}_{q,k} L_T^{-1}\lambda_T, \quad (6)$$

with bus capacitance $C_k = I_2 c_k$, $c_k \in \mathbb{R}_{>0}$, and bus conductance $G_k = I_2 g_k$, $g_k \in \mathbb{R}_{>0}$. The flux of each transmission line $k \in \mathbb{T}$ is denoted by $\lambda_{T,k} = (\lambda_{T,\alpha,k}, \lambda_{T,\beta,k}) \in \mathbb{R}^2$ and $L_{T,k} = I_2 l_{T,k}$, $l_{T,k} \in \mathbb{R}_{>0}$ denotes its inductance. For convenience of notation we define $\lambda_T = (\lambda_{T,1}, \dots, \lambda_{T,n_t}) \in \mathbb{R}^{2n_t}$ and $L_T = \mathrm{diag}(L_{T,1}, \dots, L_{T,n_t})$. The dynamics of the AC voltage bus of an inverter with index $k \in \mathbb{V}_I$ are given by

$$\dot{q}_k = -G_k C_k^{-1} q_k - L_{I,k}^{-1}\lambda_{I,k} - \mathcal{E}_{q,k} L_T^{-1}\lambda_T. \quad (7)$$

The dynamics of the load buses $k \in \mathbb{V}_l$ are given by

$$\dot{q}_k = -G_{q,k} C_k^{-1} q_k - \mathcal{E}_{q,k} L_T^{-1}\lambda_T. \quad (8)$$

The conductance $G_{q,k} = I_2 g_k(\|q_k\|)$ is used to model static resistive and more general nonlinear loads and is defined by a smooth function $g_k(s) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$.

Transmission Line Dynamics:
The dynamics of the transmission lines are given by

$$\dot{\lambda}_{T,k} = -R_{T,k} L_{T,k}^{-1}\lambda_{T,k} + \mathcal{E}_{T,k}^\top C^{-1} q, \qquad \forall k \in \mathbb{T} \quad (9)$$

where $R_{T,k} = I_2 r_{T,k}$, $r_{T,k} \in \mathbb{R}_{>0}$, is the line resistance of the k-th transmission line, $C = \mathrm{diag}(C_1, \dots, C_{n_v})$ is the capacitance matrix of the voltage buses, and $q = (q_1, \dots, q_{n_v}) \in \mathbb{R}^{2n_v}$ is the vector of voltage bus charges.

State Space Representation:
With the vectors $\theta = (\theta_1, \dots, \theta_{n_g})$, $p = (p_1, \dots, p_{n_g})$, $\lambda = (\lambda_1, \dots, \lambda_{n_g})$, $q_I = (q_{I,1}, \dots, q_{I,n_i})$, and $\lambda_I = (\lambda_{I,1}, \dots, \lambda_{I,n_i})$, the states of the overall power system model are given by $x = (\theta, p, \lambda, q_I, \lambda_I, q, \lambda_T) \in \mathbb{R}^{n_x}$, $n_x = 5n_g + 3n_i + 2n_v + 2n_t$. Using the vectors $\tau_m = (\tau_{m,1}, \dots, \tau_{m,n_g})$, $v_f = (v_{f,1}, \dots, v_{f,n_g})$, $i_{dc} = (i_{dc,1}, \dots, i_{dc,n_i})$ and $m = (m_1, \dots, m_{n_i})$, the inputs are given by $u = (\tau_m, v_f, i_{dc}, m) \in \mathbb{R}^{n_u}$, $n_u = 2n_g + 3n_i$.

Furthermore, we define the rotor speeds $\omega = M^{-1}p$ and rotor field winding currents $i_f = (i_{f,1}, \dots, i_{f,n_g})$. Finally, we let $\tau_e(\lambda, \theta) = (\tau_{e,1}, \dots, \tau_{e,n_g})$, $i_{sw}(\lambda_I, m) = (i_{sw,1}, \dots, i_{sw,n_i})$, and $v_{sw}(q_I, m) = (v_{sw,1}, \dots, v_{sw,n_i})$. To simplify the notation, we define the matrices $\mathcal{I}_f = I_{n_g} \otimes (0, 0, 1)$, as well as $\mathcal{I}_g^\top = [\mathcal{I}_s \ \mathbb{0}_{3n_g \times 2n_l + 2n_i}]$, $\mathcal{I}_I^\top = [\mathbb{0}_{2n_i \times 2n_g} \ I_{2n_i} \ \mathbb{0}_{2n_i \times 2n_l}]$, and $\mathcal{I}_s = I_{n_g} \otimes [I_2 \ \mathbb{0}_{2\times 1}]^\top$. The entire power system dynamics described by equations (3) to (9) can be compactly rewritten as $\dot{x} = f(x, u)$ with

$$\dot{x} = \begin{bmatrix} M^{-1}p \\ -DM^{-1}p - \tau_e(\lambda, \theta) + \tau_m \\ -R\mathcal{L}_\theta^{-1}\lambda + \mathcal{I}_g^\top C^{-1} q + \mathcal{I}_f v_f \\ -G_I C_I^{-1} q_I + i_{sw}(\lambda_I, m) + i_{dc} \\ -R_I L_I^{-1}\lambda_I + \mathcal{I}_I^\top C^{-1} q - v_{sw}(q_I, m) \\ -G_q C^{-1} q - \mathcal{I}_g \mathcal{L}_\theta^{-1}\lambda - \mathcal{I}_I L_I^{-1}\lambda_I - \mathcal{E} L_T^{-1}\lambda_T \\ -R_T L_T^{-1}\lambda_T + \mathcal{E}^\top C^{-1} q \end{bmatrix}, \quad (10)$$

where the matrices M, D, G_q, R, R_T, G_I, $\mathcal{L}_\theta$, L_T, L_I, C, and C_I collect the corresponding matrices of the nodes (e.g., $M = \mathrm{diag}(M_1, \dots, M_{n_g})$).

We will predominantly work with the port-Hamiltonian variables $x = (\theta, p, \lambda, \lambda_I, q_I, q, \lambda_T)$. For the sake of notational simplicity and engineering intuition we will sometimes also employ the associated *co-energy variables* $y = (\tau_e, \omega, i, i_I, v_I, v, i_T)$, where $i_I = L_I^{-1}\lambda_I$ the vector of inverter output filter currents, $v_I = C_I^{-1} q_I$ is the vector of DC voltages, $\omega = M^{-1}p$ denotes the vector of rotational frequencies, $i = \mathcal{L}_\theta^{-1}\lambda$ is the vector of stator and rotor currents, $v = C^{-1}q$ is the vector of AC voltages, and $i_T = L_T^{-1}\lambda_T$ is the vector of transmission line currents. The *co-energy variables* are depicted in Figure 1.

2.3 Desired Steady-State Behavior

We formulate the following dynamics which describe operation of the power system at a synchronous frequency $w \in \mathbb{R}$. The desired steady-state behavior (12) specifies

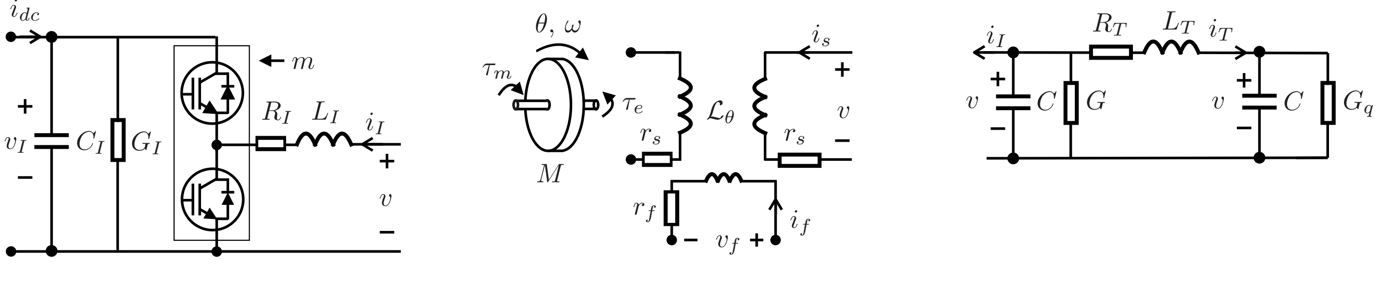

Fig. 1. Annotated diagrams of the main components of the power system: DC/AC inverter, synchronous machine, and a transmission line connecting an inverter bus and a load bus.

that DC signals are constant, and all AC signals are synchronous, balanced, and have constant amplitude.

$$\dot{\theta}_k = w, \qquad \forall k \in \mathbb{G}, \tag{11a}$$
$$\dot{p}_k = 0, \qquad \forall k \in \mathbb{G}, \tag{11b}$$
$$\dot{\lambda}_k = w\jmath\lambda_k, \qquad \forall k \in \mathbb{G}, \tag{11c}$$
$$\dot{q}_{I,k} = 0, \qquad \forall k \in \mathbb{I}, \tag{11d}$$
$$\dot{\lambda}_{I,k} = wj\lambda_{I,k}, \qquad \forall k \in \mathbb{I}, \tag{11e}$$
$$\dot{q}_k = wjq_k, \qquad \forall k \in \mathbb{V}, \tag{11f}$$
$$\dot{\lambda}_{T,k} = wj\lambda_{T,k}, \qquad \forall k \in \mathbb{T}. \tag{11g}$$

This results in the desired steady-state dynamics $\dot{x} = f_d(x, w)$ with frequency $w \in \mathbb{R}$:

$$\dot{x} = w\,(\mathbb{1}_{n_g},\, \mathbb{0}_{n_g},\, \mathcal{J}_{n_g}\lambda,\, \mathbb{0}_{n_i},\, J_{n_i}\lambda_I,\, J_{n_v}q,\, J_{n_t}\lambda_T), \tag{12}$$

where $J_n = I_n \otimes j$ and $\mathcal{J}_n = I_n \otimes \jmath$ for any $n \in \mathbb{N}_{>0}$. The steady-state behavior (11) leaves some degrees of freedom by not specifying the amplitude. In Section 3.3, we show how a specification on the amplitudes can be incorporated.

2.4 Summary of the Main Results

Loosely speaking, we show that the power system operates at the desired steady-state if and only if the following conditions are satisfied (see Theorem 2 and Theorem 4):

(1) all DC states and inputs, i.e. the DC current i_{dc} supplied to the DC/AC inverter, its DC voltage v_I, as well as the excitation voltage v_f, mechanical torque τ_m, and the rotational speed ω, need to be constant.
(2) all AC states and inputs, i.e., the modulation signal m of the inverter, the AC voltages v, transmission line currents i_T, and inverter and stator currents i_I and i_s, need to oscillate with a constant synchronous frequency ω_0 and a constant amplitude.
(3) Moreover, the nodal balance (or power flow) equations need to be satisfied.

In the next section, we define the set of states and inputs for which the power system dynamics (10) coincides with the desired dynamics (12). We derive conditions on the control inputs and synchronous frequency w which are necessary and sufficient for control-invariance of this set with respect to the nonlinear power system dynamics (10). In Section 3.3, we show that a steady-state which satisfies (12) exists if and only if there exists a corresponding solution to the nodal balance (or power flow) equations.

3. CONDITIONS FOR THE EXISTENCE OF SYNCHRONOUS STEADY-STATES

As a starting point for our analysis, we define the set $\mathcal{S}$ on which the the dynamics (10) and (12) coincide and the residual dynamics $\rho(x, u, w) = f(x, u) - f_d(x, w)$ vanish:

$$\mathcal{S} := \{(x, u, w) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R} \,|\, \rho(x, u, w) = \mathbb{0}_{n_x}\}. \tag{13}$$

Operation of the power system at the desired steady-state requires that all trajectories of the dynamics (10) starting in $\mathcal{S}$ remain in $\mathcal{S}$ for all time, i.e. the set $\mathcal{S}$ needs to be control-invariant with respect to the dynamics (10) and the input u.

3.1 Control-Invariance of the Set $\mathcal{S}$

To establish control-invariance of $\mathcal{S}$, we consider the dynamics obtained by combining the nonlinear power grid dynamics described by $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ with controller (generator and inverter) dynamics $g_u : \mathbb{R}_{>0} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R} \to \mathbb{R}^{n_u}$ and $g_w : \mathbb{R} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R} \to \mathbb{R}$ describing the dynamics of the target frequency possibly both depending on time and the system state:

$$(\dot{x}, \dot{u}, \dot{w}) = (f(x, u),\, g_u(t, x, u, w),\, g_w(t, x, u, w)). \tag{14}$$

Theorem 2. **(Control-invariance conditon)** The set $\mathcal{S}$ is invariant with respect to the dynamics (14) if and only if $g_w(t, x, u, w) = 0$ and $g_u(t, x, u, w) = (\mathbb{0}_{2n_g+n_i}, wJ_{n_i}m)$ for all $(x, u, w) \in \mathcal{S}$. In other words, the control inputs τ_m, v_f and i_{dc} as well as the synchronous frequency w take constant values on $\mathcal{S}$, and the control input m_k satisfies $\dot{m}_k = wjm_k$ on $\mathcal{S}$.

Proof. $\mathcal{S}$ is invariant with respect to (14) if and only if $\frac{d}{dt}\rho(x, u, w) = 0$ for all $(x, u, w) \in \mathcal{S}$ or, equivalently, if and only if the tangency condition $(\dot{x}, \dot{u}, \dot{w}) \in \ker\{\nabla_{(x,u,w)}\rho(x, u, w)^\top\}$ holds for all $(x, u, w) \in \mathcal{S}$. On $\mathcal{S}$, it holds that $f_d(x, w) = f(x, u)$, resulting in the following necessary and sufficient condition for invariance of $\mathcal{S}$ for all $(x, u, w) \in \mathcal{S}$:

$$\frac{\partial f}{\partial x} f_d - \frac{\partial f_d}{\partial x} f_d + \frac{\partial f}{\partial u} g_u + \frac{\partial f_d}{\partial w} g_w = 0. \tag{15}$$

Using $g_u := (g_\tau, g_{v_f}, g_{i_{dc}}, g_m)$ one obtains:

$$\frac{\partial f}{\partial u} g_u = \begin{bmatrix} \mathbb{0}_{n_g} \\ g_\tau \\ \mathcal{I}_f g_{v_f} \\ g_{i_{dc}} + \dfrac{\partial i_{sw}}{\partial m} g_m \\ -\dfrac{\partial v_{sw}}{\partial m} g_m \\ \mathbb{0}_{2n_v} \\ \mathbb{0}_{2n_t} \end{bmatrix}, \quad \frac{\partial f_d}{\partial w} g_w = g_w \begin{bmatrix} \mathbb{1}_{n_g} \\ \mathbb{0}_{n_g} \\ \mathcal{J}_{n_g}\lambda \\ \mathbb{0}_{n_i} \\ J_{n_i}\lambda_I \\ J_{n_v} q \\ J_{n_t}\lambda_T \end{bmatrix}, \tag{16}$$

$$\frac{\partial f_d}{\partial x} f_d = (\mathbb{0}_{2n_g}, w^2\mathcal{J}_{n_g}^2\lambda, \mathbb{0}_{n_i}, w^2 J_{n_i}^2\lambda_I, w^2 J_{n_v}^2 q, w^2 J_{n_t}^2\lambda_T). \tag{17}$$

To compute the product $\Xi = \frac{\partial f}{\partial x} f_d$ we require the following preliminary step. Consider for now the derivative $\frac{d}{dt} G_{q,k} q_k$ with respect to the dynamics $\dot{q}_k = \omega_0 j q_k$. With $G_{q,k} = I_2 g_k(\|q_k\|)$ one obtains $\frac{d}{dt} G_{q,k} q_k = I_2 \left(\frac{d}{dt} g_k(\|q_k\|)\right) q_k + G_{q,k} j q_k$. Because $\frac{d}{dt}\|q_k\| = 0$ holds for $\dot{q}_k = \omega_0 j q_k$, it immediately follows that $\frac{d}{dt} g_k(\|q_k\|) = 0$ holds along the vector field $\dot{q}_k = \omega_0 j q_k$. This fact together with $\frac{d}{dt} G_k(\|q_k\|) q_k = \left(\frac{\partial}{\partial q_k} G_{q,k} q_k\right) j q_k$ and $C_k^{-1} = I_2 c_k^{-1}$ results in:

$$\left(\frac{\partial}{\partial q_k} G_{q,k} C_k^{-1} q_k\right) j q_k = G_{q,k} C_k^{-1} j q_k. \tag{18}$$

By using (18), the partial derivatives $\frac{\partial \tau_{e,k}}{\partial \theta} = \lambda_k^\top (\mathcal{L}_{\theta,k}^{-1}\jmath^\top + \jmath\mathcal{L}_{\theta,k}^{-1})\jmath^\top \lambda_k$, $\frac{\partial \tau_{e,k}}{\partial \lambda} = \lambda_k^\top (\mathcal{L}_{\theta,k}^{-1}\jmath^\top + \jmath\mathcal{L}_{\theta,k}^{-1})$, as well as $\mathcal{J}_{n_g} = -\mathcal{J}_{n_g}^\top$, $\mathcal{I}_g^\top J_{n_v} = \mathcal{J}_{n_g}\mathcal{I}_g^\top$, $J_{n_v}\mathcal{I}_g = \mathcal{I}_g\mathcal{J}_{n_g}$, $\mathcal{E}J_{n_t} = J_{n_v}\mathcal{E}$ one obtains:

$$\Xi = w \begin{bmatrix} \mathbb{0}_{2n_g} \\ \mathcal{J}_{n_g}(-R\mathcal{L}_\theta^{-1}\lambda + \mathcal{I}_g^\top C^{-1} q) \\ \frac{1}{2}\operatorname{diag}(m_k^\top) L_I^{-1} J_{n_i}\lambda_I \\ J_{n_i}(-R_I L_I^{-1}\lambda_I + \mathcal{I}_I^\top C^{-1} q) \\ J_{n_v}(-G_q C^{-1} q - \mathcal{I}_g\mathcal{L}_\theta^{-1}\lambda - \mathcal{I}_I\mathcal{L}_I^{-1}\lambda_I - \mathcal{E}L_T^{-1}\lambda_T) \\ J_{n_t}(-R_T L_T^{-1}\lambda_T + \mathcal{E}^\top C^{-1} q) \end{bmatrix}.$$

By definition of $\mathcal{S}$ we have for all $(x,u,w) \in \mathcal{S}$ that:

$$\Xi = \begin{bmatrix} \mathbb{0}_{2n_g} \\ w\mathcal{J}_{n_g}(w\mathcal{J}_{n_g}\lambda - \mathcal{I}_f v_f) \\ w\frac{1}{2}\operatorname{diag}(m_k^\top)L_I^{-1}J_{n_i}\lambda_I \\ wJ_{n_i}(wJ_{n_i}\lambda_I + v_{sw}(q_I,m)) \\ w^2 J_{n_v}^2 q \\ w^2 J_{n_t}^2 \lambda_T \end{bmatrix}. \tag{19}$$

Moreover, using $\mathcal{J}_{n_g}\mathcal{I}_f = 0$ as well as

$$m_k^\top L_{I,k}^{-1} wj\lambda_{I,k} = -\lambda_{I,k}^\top L_{I,k}^{-1} wjm_k = -\frac{\partial i_{sw,k}}{\partial m_k} wjm_k,$$

$$wjv_{sw,k} = C_{I,k}^{-1} q_{I,k} wjm_k = \frac{\partial v_{sw,k}}{\partial m_k} wjm_k,$$

it can be verified that:

$$\Xi = \begin{bmatrix} \mathbb{0}_{2n_g} \\ w^2\mathcal{J}_{n_g}^2\lambda \\ -\frac{\partial i_{sw}}{\partial m} wJ_{n_i} m \\ w^2 J_{n_i}^2 \lambda_I + \frac{\partial v_{sw}}{\partial m} wJ_{n_i} m \\ w^2 J_{n_v}^2 q \\ w^2 J_{n_t}^2 \lambda_T \end{bmatrix}, \tag{20}$$

By substituting (16), (17), and (20) into (15) one obtains the following condition for invariance of $\mathcal{S}$ with respect to the dynamics (14) for all $(x,u,w) \in \mathcal{S}$:

$$\begin{bmatrix} \mathbb{0}_{n_g} \\ \mathbb{0}_{n_g} \\ \mathbb{0}_{3n_g} \\ -\frac{\partial i_{sw}}{\partial m} wJ_{n_i} m \\ \frac{\partial v_{sw}}{\partial m} wJ_{n_i} m \\ \mathbb{0}_{2n_v} \\ \mathbb{0}_{2n_t} \end{bmatrix} + \begin{bmatrix} \mathbb{0}_{n_g} \\ g_\tau \\ \mathcal{I}_f g_{v_f} \\ g_{i_{dc}} + \frac{\partial i_{sw}}{\partial m} g_m \\ -\frac{\partial v_{sw}}{\partial m} g_m \\ \mathbb{0}_{2n_v} \\ \mathbb{0}_{2n_t} \end{bmatrix} + g_w \begin{bmatrix} \mathbb{1}_{n_g} \\ \mathbb{0}_{n_g} \\ \mathcal{J}_{n_g}\lambda \\ \mathbb{0}_{n_i} \\ J_{n_i}\lambda_I \\ J_{n_v} q \\ J_{n_t}\lambda_T \end{bmatrix} = \mathbb{0}_{n_x}. \tag{21}$$

The first two lines imply $g_w = 0$ and $g_\tau = 0$. Moreover, letting $g_w = 0$ the third and fifth line imply that $g_{v_f} = 0$ and $g_m = wJ_{n_i}m$. Finally, letting $g_w = 0$ and $g_m = wJ_{n_i}m$ the fourth line holds if and only if $g_{i_{dc}} = 0$. Thus, (21) holds if and only if $g_w(t,x,u,w) = 0$ and $g_u(t,x,u,w) = (\mathbb{0}_{2n_g+n_i}, wJ_{n_i}m)$ for all $(x,u,w) \in \mathcal{S}$. □

Theorem 2 shows that the power system admits only steady-states with a *constant* synchronous frequency $w(t)$. Moreover, it shows that on $\mathcal{S}$ the DC current i_{dc} and all the generator inputs need to be constant and that the duty cycle m needs to oscillate with the synchronous frequency w. These conditions can be seen as necessary conditions for controllers in steady-state, i.e. every controller used to stabilize the desired steady-state of the power system has to satisfy these conditions in steady-state. In particular, this reveals that a controller of the DC/AC inverter requires an internal oscillator model, which is equal to $\dot{m}_k = \omega_0 jm_k$ on $\mathcal{S}$, to operate the power system at the desired steady-state. This last control specification, derived here constructively, is implicit in the standard droop control (already formulated in polar coordinates), virtual oscillator control, and generator emulation or matching control schemes.

Based on this result we restrict ourselves from now to a constant (possibly zero) frequency $\omega_0 \in \mathbb{R}$ corresponding to the nominal synchronous operating frequency of the power system. Letting $w = \omega_0 \in \mathbb{R}$ for all $t \in \mathbb{R}_{\geq 0}$ results in the following set parametrized in $\omega_0 \in \mathbb{R}$:

$$\mathcal{S}_{\omega_0} := \{(x,u) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \mid \rho(x,u,\omega_0) = \mathbb{0}_{n_x}\}. \tag{22}$$

Furthermore, $\rho(x,u,\omega_0)$ can be rewritten as follows:

$$\rho(x,u,\omega_0) = \begin{bmatrix} M^{-1}p - \mathbb{1}_{n_g}\omega_0 \\ DM^{-1}p + \tau_e(\lambda,\theta) - \tau_m \\ (R\mathcal{L}_\theta^{-1} + \omega_0\mathcal{J}_{n_g})\lambda - \mathcal{I}_g^\top C^{-1}q - \mathcal{I}_f v_f \\ G_I C_I^{-1} q_I - i_{sw}(\lambda_I, m) - i_{dc} \\ (RL_I^{-1} + \omega_0 J_{n_i})\lambda_I - I_I^\top C^{-1} q + v_{sw}(q_I, m) \\ (G_q C^{-1} + \omega_0 J_{n_v})q + \mathcal{I}_g\mathcal{L}_\theta^{-1}\lambda + \mathcal{I}_I L_I^{-1}\lambda_I + \mathcal{E}L_T^{-1}\lambda_T \\ (R_T L_T^{-1} + \omega_0 J_{n_t})\lambda_T - \mathcal{E}^\top C^{-1} q \end{bmatrix}.$$

3.2 Steady-State Separation of Network and Sources

In the following, we establish a connection between the steady-state conditions (22) and balance equations of the network (often formulated as power flow equations). The overall power system (10) can be partitioned into n_i sources with dynamics (1), n_g sources with dynamics (3), and the network consisting of the transmission line and voltage bus dynamics. Each source is interconnected to the network by the voltage $v_k = C_k^{-1} q_k \in \mathbb{R}^2$ of its AC terminal (output of the network) and by the currents at the inverter and generator terminals, $i_{I,k} = L_{I,k}^{-1}\lambda_{I,k} \in \mathbb{R}^2$ and $i_{s,k} = [I_2 \ \mathbb{0}_{2\times 1}]\mathcal{L}_{\theta,k}^{-1}\lambda_k \in \mathbb{R}^2$, injected into the network.

In the following, we separate the steady-state conditions (22) of sources and the network and show that the steady-state conditions for the network can be used to characterize the desired steady-state of the full system.

For notational convenience we define the vector of inverter output filter currents $i_I = (i_{I,1}, \ldots, i_{I,n_i})$ and voltages $\nu_{I,k} \in \mathbb{R}^2$ for $k \in \mathbb{V}_I$:

$$i_I = L_I^{-1}\lambda_I, \tag{23}$$

$$\nu_{I,k} = v_k - Z_{I,k} i_{I,k},, \tag{24}$$

where $Z_{I,k} = R_{I,k} + \omega_0 jL_{I,k}$ is the output filter impedance.

For each generator with index $k \in \mathbb{V}_g$ we define vector of stator currents $i_s = (i_{s,1}, \ldots, i_{s,n_g})$ and the vector of the back-EMF voltages $\nu_k = (\nu_{\alpha,k}, \nu_{\beta,k})$:

$$i_s = \mathcal{I}_s \mathcal{L}_\theta^{-1}\lambda, \tag{25}$$

$$\nu_k = v_k - Z_{s,k} i_{s,k}, \tag{26}$$

where $Z_{s,k} = R_{s,k} + \omega_0 jL_{s,k}$ is the stator impedance.

The following equations describe Kirchhoff's current law at each AC voltage bus, as well as Kirchhoff's voltage law over the network branches:

$$\rho_N(i_s, i_I, v, i_T) = \begin{bmatrix} (G_v + \omega_0 J_{n_v} C)v + (i_s, i_{dc}, \mathbb{0}_{2n_l}) + \mathcal{E}i_T \\ (R_T + \omega_0 J_{n_t} L_T)i_T - \mathcal{E}^\top v \end{bmatrix},$$

with the conductance $G_{v,k} = I_2 g_k(\|C_k v_k\|)$. Based on the vector of current injections and transmission network states $(i_s, i_I, v, i_T) \in \mathbb{R}^{n_z}$ we define the solution set $\mathcal{N}_{\omega_0}$ of the transmission network equations:

$$\mathcal{N}_{\omega_0} := \{(i_s, i_I, v, i_T) \in \mathbb{R}^{n_z} \mid \rho_N(i_s, i_I, v, i_T) = \mathbb{0}_{n_N}\}. \tag{27}$$

The steady-state equations of the transmission network do not uniquely specify the steady-state of the sources. For each generator, the steady-state is specified up to the polarization of the rotor, i.e., the sign of the excitation voltage $v_{f,k}$, and the steady-state of the inverters allows for a trade-off between the magnitude of the modulation input m_k and the DC voltage $v_{I,k}$. Hence, to characterize all steady states, we introduce the rotor polarization $\sigma_k \in \{-1, 1\}$ and the nominal amplitude $\hat{m}_k \in \mathbb{R}_{(0,1)}$ of the modulation signal m_k.

The following statement extends the result in Groß et al. (2016) to the power system model including DC/AC inverters. It formalizes the separation between the network and the sources and establishes that the full system state can be recovered from a solution to the steady-state equations (27) of the transmission network.

Theorem 3. **(Network source separation)** Consider the sets $\mathcal{N}_{\omega_0}$ and $\mathcal{S}_{\omega_0}$ defined in (22) and (27). The following statements are equivalent:

(1) There exists $(i_s, i_I, v, i_T) \in \mathcal{N}_{\omega_0}$,
(2) There exists $(\theta, p, \lambda, q_I, \lambda_I, q, \lambda_T, u) \in \mathcal{S}_{\omega_0}$ such that $v_{I,k} \in \mathbb{R}_{\geq 0}$ and $\|m_k\| \leq 1$.

Moreover, for any $\omega_0 \neq 0$ and every $(i_s, i_I, v, i_T) \in \mathcal{N}_{\omega_0}$ all corresponding $(\theta, p, \lambda, q_I, \lambda_I, q, \lambda_T, u) \in \mathcal{S}_{\omega_0}$ with $v_{I,k} \in \mathbb{R}_{\geq 0}$ and $\|m_k\| \leq 1$ are explicitly given by the nominal amplitude of the modulation $\hat{m}_k \in \mathbb{R}_{(0,1)}$ and:

$$v_{I,k} = \frac{2}{\hat{m}_k} \|\nu_{I,k}\|, \tag{28a}$$

$$m v_{I,k} = 2\nu_{I,k} \tag{28b}$$

as well as the rotor polarization $\sigma_k \in \{-1, 1\}$ and:

$$i_{f,k} = \sigma_k \omega_0^{-1} l_{m,k}^{-1} \|\nu_k\|, \tag{28c}$$

$$\mathcal{R}_{\theta_k}(0, \|\nu_k\|) = \sigma_k \nu_k, \tag{28d}$$

and

$$v_{f,k} = r_{f,k} i_{f,k}, \tag{28e}$$

$$\tau_{m,k} = D_k \omega_0 + \tau_{e,k}(\lambda_k, \theta_k), \tag{28f}$$

$$i_{dc,k} = G_{I,k} v_{I,k} + \tfrac{1}{2} i_{I,k}^\top m_k, \tag{28g}$$

and $\lambda_k = \mathcal{L}_{\theta,k} i_k$, $p_k = M_k \omega_0$, $q = Cv$, $\lambda_T = L_T i_T$, and $\tau_{e,k}(\lambda_k, \theta_k) = \frac{1}{2} \lambda_k^\top (\mathcal{L}_{\theta,k}^{-1} \mathcal{J}^\top + \mathcal{J} \mathcal{L}_{\theta,k}^{-1}) \lambda_k$.

Proof. We first establish that the second statement implies the first statement. By definition of the sets $\mathcal{N}_{\omega_0}$, $\mathcal{V}_\mathcal{N}$, $\mathcal{S}_{\omega_0}$, and $\mathcal{V}$, and with the identity (25) it directly follows that $(\mathcal{I}_s \mathcal{L}_\theta^{-1} \lambda, L_I^{-1} \lambda_I, C^{-1} q, L_T^{-1} \lambda_T) \in \mathcal{N}_{\omega_0}$ holds for all $(\theta, p, \lambda, q_I, \lambda_I, q, \lambda_T, u) \in \mathcal{S}_{\omega_0}$.

To establish that the first statement implies the second statement, note that $(i_s, i_I, v, i_T) \in \mathcal{V}_\mathcal{N}$ implies $(x, u) \in \mathcal{V}$. Moreover, $(\theta, p, \lambda, q_I, \lambda_I, q, \lambda_T, u) \in \mathcal{S}_{\omega_0}$ requires that the following equation holds:

$$(R L_I^{-1} + \omega_0 J_{n_i}) \lambda_I - I_I^\top C^{-1} q + v_{sw}(q_I, m) = \mathbb{0}_{2n_i}. \tag{29}$$

By using $\lambda_I = L_I i_I$, $v = C^{-1} q$, $v_I C_I^{-1} q_I$, and considering the components of (29), we obtain the condition $-\nu_{I,k} + \frac{1}{2} v_{I,k} m_k = \mathbb{0}_2$ for all $k \in \mathbb{I}$. This results in $v_{I,k} m_k = 2\nu_{I,k}$. In addition, $\hat{m}_k \in \mathbb{R}_{>0}$ and (28a) parametrize all m_k and $v_{I,k}$ for which $\|m_k\| \leq 1$ and $v_{I,k} \in \mathbb{R}_{\geq 0}$ hold. Moreover, (28g) is directly obtained from the following condition:

$$G_I C_I^{-1} q_I - \tfrac{1}{2} \operatorname{diag}(\lambda_{I,k}^\top L_{I,k}^{-1}) m - i_{dc} = \mathbb{0}_{n_i}. \tag{30}$$

Next, steady-state operation of generator $k \in \mathbb{G}$ requires:

$$\mathbb{0}_{3n_g} = (R \mathcal{L}_\theta^{-1} + \omega_0 \mathcal{J}_{n_g}) \lambda - \mathcal{I}_g^\top C^{-1} q - \mathcal{I}_f v_f. \tag{31}$$

By using $\lambda_k = \mathcal{L}_{\theta,k} i_k$, $v_k = C_k^{-1} q_k$, and considering the components of (31), we obtain the following condition:

$$\begin{bmatrix} R_{s,k} + \omega_0 j L_{s,k} & \omega_0 j \mathcal{R}_{\theta_k} L_{m,k} \\ 0 & r_{f,k} \end{bmatrix} \begin{bmatrix} i_{s,k} \\ i_{f,k} \end{bmatrix} - \begin{bmatrix} v_k \\ v_{f,k} \end{bmatrix} = 0.$$

This holds if and only if $v_{f,k} = r_{f,k} i_{f,k}$. If we use $j \mathcal{R}_{\theta_k} = \mathcal{R}_{\theta_k} j$ this results in:

$$j \mathcal{R}_{\theta_k} L_{m,k} \omega_0 i_{f,k} = \mathcal{R}_{\theta_k} \begin{bmatrix} 0 \\ l_{m,k} \omega_0 i_{f,k} \end{bmatrix} = \nu_k. \tag{32}$$

It can be seen, that there exists θ_k such that (32) holds if and only if $\|\nu_k\| = \|l_{m,k} \omega_0 i_{f,k}\|$, i.e., if and only if (28c) holds for $\sigma \in \{-1, 1\}$. Next, consider the case $\omega_0 \neq 0$. Using the current $i_{f,k} = \omega_0^{-1} l_{m,k}^{-1} \|\nu_k\|$ results in the condition $\mathcal{R}_{\theta_k}(0, \|\nu_k\|) = \nu_k$, for all $k \in \mathbb{G}$ whereas $i_{f,k} = -\omega_0^{-1} l_{m,k}^{-1} \|\nu_k\|$ results in $\mathcal{R}_{\theta_k}(0, \|\nu_k\|) = -\nu_k$ for all $k \in \mathbb{G}$. Moreover, the case $\omega_0 = 0$ requires $\nu_k = 0$ and (32) holds for any θ_k and any $i_{f,k}$. In either case, $\lambda_k = \mathcal{L}_{\theta,k} i_k$ and $\tau_{e,k}(\lambda_k, \theta_k) = \frac{1}{2} \lambda_k^\top (\mathcal{L}_{\theta,k}^{-1} \mathcal{J}^\top + \mathcal{J} \mathcal{L}_{\theta,k}^{-1}) \lambda_k$ can be explicitly recovered. Moreover, the following conditions need to hold:

$$\mathbb{0}_{n_g} = M^{-1} p - \mathbb{1}_{n_g} \omega_0, \tag{33}$$

$$\mathbb{0}_{n_g} = D M^{-1} p + \tau_e(\lambda, \theta) - \tau_m. \tag{34}$$

These equations hold if and only if $M_k^{-1} p_k = \omega_0$ and if and only if $\tau_{m,k} = D_k \omega_0 + \tau_{e_k}$. It directly follows that there exist $(\theta, p, \lambda, q, \lambda_T, u) \in \mathcal{S}_{\omega_0}$ if there exist $(i_s, v, i_T) \in \mathcal{N}_{\omega_0}$. Finally, the above shows that the conditions (28) are also necessary for $(x, u) \in \mathcal{S}_{\omega_0}$. □

Broadly speaking, Theorem 3 shows that the conditions for operation of the power system (10) at the desired steady-state can be fully expressed in terms of the transmission network equations and the currents injected into the network at the converter and generator terminals. This results in a simpler condition for the existence of a steady state which no longer depends on the nonlinear dynamics of the synchronous machines or DC/AC inverters.

Hence, we can compute all states and inputs (x, u) such that $(x, u) \in \mathcal{S}_{\omega_0}$ holds based on a solution to the transmission network equations $\rho_N(i_s, v, i_T) = \mathbb{0}_{n_N}$ satisfying $(i_s, v, i_T) \in \mathcal{V}_\mathcal{N}$. In particular, for each solution $\rho_N(i_s, v, i_T) = \mathbb{0}_{n_N}$ of the network equations one can directly compute ν_k for all $k \in \mathbb{G}$ and $\nu_{I,k}$ for all $k \in \mathbb{I}$. If $\nu_{I,k} \neq 0$ the modulation signal m_k, as well as the steady-state voltage of the DC-link capacitor $v_{I,k}$ are given by (28a) and (28b) and $\hat{m}_k$. Similarly, if $\nu_k \neq \mathbb{0}$ and $\omega_0 \neq 0$, the steady-state field winding currents $i_{f,k}$ and angles θ_k are given by (28c), (28d) and $\sigma \in \{-1, 1\}$; see also (Groß et al., 2016) for the detailed calculations for generators.

The conditions given in Theorem 3 ensure that $(x, u) \in \mathcal{S}_{\omega_0}$ holds point-wise in time and the conditions of Theorem 2 ensure invariance of $\mathcal{S}_{\omega_0}$. By using the same arguments used in Groß et al. (2016) it can be shown that the specifications are compatible, i.e. the inputs defined by Theorem 3 satisfy the conditions of Theorem 2.

3.3 Nodal Balance Equations

The aim of this section is to derive conditions for the existence of a steady-state behavior that satisfies the specifications of the desired steady-state. We show that the nodal current balance equations equivalently characterize the set $\mathcal{S}_{\omega_0}$, i.e. the power system admits the desired steady-state behavior if and only if the nodal balance equations have a solution.

The set $\mathcal{P}_{\omega_0}$ describes the solutions of the nodal current balance equations (Sauer and Pai, 1998):

$$\mathcal{P}_{\omega_0} := \{(i_s, i_I, v) \in \mathbb{R}^{n_p} \,|(i_s, i_{dc}, \mathbb{0}_{2n_l}) + Y_N v = \mathbb{0}_{n_K}\}\,.$$

Here $Y_v = G_v + \omega_0 J_{n_v} C$ is the matrix of shunt load admittances, $Z_T = R_T + \omega_0 J_{n_t} L_T$ is the (invertible) matrix of branch impedances, and $Y_N = Y_v + \mathcal{E} Z_T^{-1} \mathcal{E}^\top$ is the network admittance matrix. The next result shows that for each $(i_s, i_I, v) \in \mathcal{P}_{\omega_0}$, there exists a steady-state of the overall power system – and vice versa.

Theorem 4. **(Nodal current balance equations)** The following statements are equivalent:

(1) There exists $(i_s, i_I, v) \in \mathcal{P}_{\omega_0}$,
(2) There exists $(\theta, p, \lambda, q_I, \lambda_I, q, \lambda_T, u) \in \mathcal{S}_{\omega_0}$.

A proof can be found in Arghir et al. (2016).

In practice, the amplitude of the AC voltages in steady-state is required to be near the (non-zero) nominal voltage amplitude of the power system. Theorem 4 implies that the voltage magnitude of any solution to the nodal balance equations and the voltage magnitude of the corresponding steady-state behavior are identical. In other words, the steady-state specification (11) in conjunction with a suitable solution to the power flow equations (usually obtained by constrained optimal generation dispatch) fully specifies the steady-state behavior.

4. CONCLUSION

In this paper, we provided results on the steady-state behavior of power systems modeled by nonlinear DC/AC inverter dynamics, nonlinear synchronous machine dynamics, a dynamic model of the transmission network, and nonlinear static loads. In particular, the first-principles model contains averaged dynamics of DC/AC inverters, including the charge dynamics of the DC-link capacitor, and is suitable to analyze power systems with low or no rotational inertia.

The steady-state behavior considered in this work is defined by balanced and sinusoidal three-phase AC signals of the same synchronous frequency. Building on our previous results (Arghir et al., 2016), we show that the power system admits such a steady-state behavior if the network balance equations admit a corresponding solution. We derive necessary and sufficient conditions for the inputs in steady-state. In particular, in steady-state, the current supplied to the DC-link capacitor of the DC/AC converter as well as the control inputs of the generators are required to be constant. Moreover, the modulation signal of each inverter is required to rotate with the synchronous frequency. This analysis directly reveals necessary conditions that any controller needs to satisfy in steady-state.

In the literature on multi-machine power systems, it is often assumed a priori that the admits such a steady-state behavior if the generator controls are constant and the network balance equations can be solved. Moreover, heuristic control strategies for inverters such as droop control, virtual oscillator control, synchronverters, generator emulation, and matching control, implicitly satisfy our steady-state control specifications and therefore admit the desired steady-state behavior. We show that all of these conditions can be constructively obtained from first-principle and are, in fact, necessary and sufficient for the power system to admit the desired steady-state behavior.

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