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Robust and Optimal Control: a Convex Approach

Lecture Notes
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Preliminary version
Please send comments/suggestions, and
report all errors/typos.

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Contents

1 Introduction

Course prerequisites:

- Linear systems, e.g.: $Ax = b$
- Differential equations: $\dot{x} = f(x)$
- Continuous unconstrained optimization: $\min f(x) \longrightarrow \frac{\partial f}{\partial x} = 0$
- Linear programming: $\begin{array}{l} \min c^T x \\ \text{s.t.} Ax = b \end{array}$
- Matrix algebra, e.g.: $AX = B \longrightarrow X = A^{-1}B$

Examples of optimization problems:

- What is the fastest way to get from A to B?
- Where is the optimal launch point into the orbit?
- How does the fuel-optimal trajectory to Jupiter look like?
- Optimal rate of infusion.
- Stock market: stochastic optimal control

A basic classification of optimization problems is shown in Table 1. In this course we will focus mainly on *optimal control*.

Table 1: Classification of optimization problems

	One decision maker	Several decision makers
No time	Static optimization	Game theory
Dynamic	Optimal control	Dynamic games

Example 1.1: Classify the following activities:

- Parking a car: optimal control.
- Drive to Rome: dynamic games.
- Buy Christmas gifts: static optimization.
- Buy operating system: game theory.

□

Basic Notions:

Robustness. We need to make sure that our solution is not sensitive to changes (perturbations) in the model.

Convexity. We need mathematical conditions to ensure that solutions are optimal; under convexity assumptions, there exist efficient ways for computing the optimal solution.

2 Static Optimization

2.1 Static Optimization without Constraints

Univariate case. For a scalar function $f : \mathbb{R} \to \mathbb{R}$, a *necessary* condition (first order condition) for x^o being a local minimum of f at x^o is given by

$$f_x(x^o) = \left.\frac{\partial f(x)}{\partial x}\right|_{x^o} = 0.$$

The following conditions are *sufficient* for local minimum at x^o (second order condition):

$$\left.\frac{\partial f(x)}{\partial x}\right|_{x^o} = 0 \quad \text{and} \quad \left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x^o} > 0.$$

Example 2.1: Find the minimum of $f(x) = x^4 + 2x^3$. Figure 2.1 shows a plot of $f(x)$.

$$\frac{\partial f(x)}{\partial x} = 4x^3 + 16x^2 \overset{!}{=} 0 \quad \Longrightarrow \quad x = 0, 0, -\tfrac{3}{2}$$

$$\frac{\partial f^2(x)}{\partial x^2} = 12x^2 + 12x\Big|_{x=-\frac{3}{2}} = 9 > 0 \quad \Longrightarrow \quad \text{minimum.}$$

$\square$

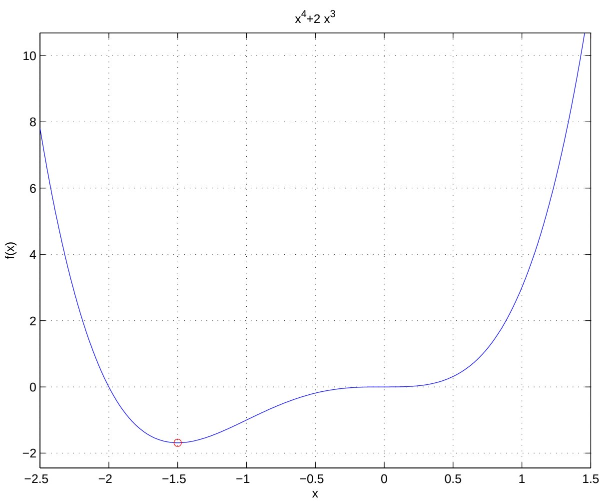

Figure 2.1: Function $f(x) = x^4 + 2x^3$. The circle denotes the minimum.

Multivariate case. For a scalar function $f : \mathbb{R}^n \to \mathbb{R}$, a *necessary* condition for x^o being a local minimum of f at x^o is given by

$$\left.\frac{\partial f(x)}{\partial x}\right|_{x^o} = f_x(x^o) = \nabla f(x^o) = \begin{bmatrix} \left.\frac{\partial f(x)}{\partial x_1}\right|_{x^o} \\ \vdots \\ \left.\frac{\partial f(x)}{\partial x_n}\right|_{x^o} \end{bmatrix} = 0$$

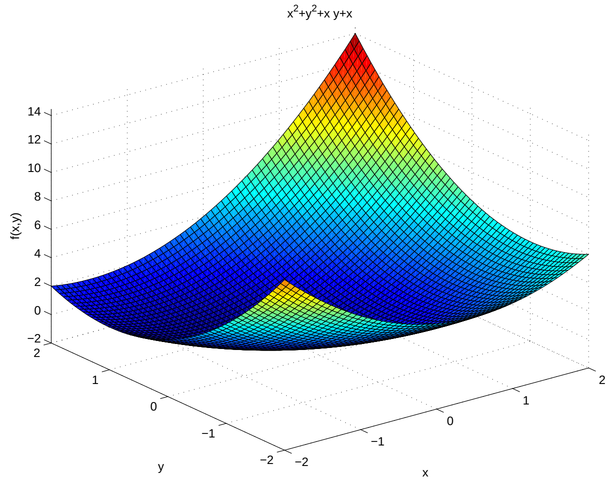

Figure 2.2: Function $f(x,y) = x^2 + y^2 + xy + x$.

A *sufficient* condition for a candidate point x^o being a (local) minimum, is a *positive definite*[1] Hessian matrix at x^o:

$$\left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x^o} = f_{xx}(x^o) = \Delta f(x^o) = \begin{bmatrix} \left.\frac{\partial^2 f(x)}{\partial x_1 \partial x_1}\right|_{x^o} & \cdots & \left.\frac{\partial^2 f(x)}{\partial x_1 \partial x_n}\right|_{x^o} \\ \vdots & & \vdots \\ \left.\frac{\partial^2 f(x)}{\partial x_n \partial x_1}\right|_{x^o} & \cdots & \left.\frac{\partial^2 f(x)}{\partial x_n \partial x_n}\right|_{x^o} \end{bmatrix} \succ 0.$$

This can be easily shown by considering a truncated local Taylor approximation of $f(x)$;

$$df = \underbrace{\left(\left.\frac{\partial f(x)}{\partial x}\right|_{x^o}\right)^T}_{0} dx + \frac{1}{2} dx^T \left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x^o} dx + \ldots$$

If x^o is a local minimum, df must be positive for any variation dx. Thus

$$dx^T \left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x^o} dx > 0 \quad \forall\, dx \neq 0.$$

Example 2.2: Find the minimum of the given $f(x,y) = x^2 + y^2 + xy + x$ which is illustrated by Fig. 2.2.

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} 2x + y + 1 \\ x + 2y \end{bmatrix} = 0 \quad \Longrightarrow \quad x = -\tfrac{2}{7},\ y = \tfrac{1}{7}$$

$$\frac{\partial^2 f(x)}{\partial x^2} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\det[2] = 2 > 0\,, \quad \det \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 3 > 0 \quad \Longrightarrow \quad \text{minimum.}$$

□

[1] A matrix A is called *positive semidefinite* if $x^T A x \geq 0$ for all $x \in \mathbb{R}^n$ and *positive definite* if $x^T A x > 0$ for all $x \in \mathbb{R}^n, x \neq 0$. For convenience, we will write $A \succeq 0$ and $A \succ 0$ respectively.

2.2 Static Optimization with Equality Constraints

Consider the following optimization problem:

$$\begin{aligned} &\min L(x, u) \\ &\text{s.t.} \\ &f_i(x, u) = 0 \qquad i = 1, \ldots, n \end{aligned}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, i.e. the number of equality constraints equals the dimension of x.

First Order Conditions. The variations of the objective function L and constraint vector f with respect to x and u are as follows:

$$dL = L_x^T dx + L_u^T du \tag{2.1}$$

$$df = f_x dx + f_u du. \tag{2.2}$$

Since the constraints have to hold for small variations, equation (2.2) has to equal zero yielding *compatibility conditions* for dx:

$$dx = -f_x^{-1} f_u du. \tag{2.3}$$

Inserting (2.3) back into (2.1) yields dL with the constraints considered:

$$dL = (-L_x^T f_x^{-1} f_u + L_u^T) du$$

For a local minimum, dL must be ≥ 0 for arbitrary variations du. Thus the first *first order conditions* are given by:

$$L_u^T - L_x^T f_x^{-1} f_u = 0. \tag{2.4}$$

Lagrange Multiplier Method. As already mentioned above, for an optimum, equations (2.1) and (2.2) must be equal to zero, i.e.:

$$\begin{bmatrix} dL \\ df \end{bmatrix} = \begin{bmatrix} L_x^T & L_u^T \\ f_x & f_u \end{bmatrix} \begin{bmatrix} dx \\ du \end{bmatrix} = 0. \tag{2.5}$$

In order for (2.5) having a non-trivial solution for $[dx^T \; du^T]$, the rank of the matrix must be less than $n + 1$. Hence, its rows must be linearly dependent:

$$\begin{bmatrix} 1 & \lambda^T \end{bmatrix} \begin{bmatrix} L_x^T & L_u^T \\ f_x & f_u \end{bmatrix} = 0, \tag{2.6}$$

where λ denotes the vector of the so-called *Lagrange multipliers*. In other words, the gradient of the objective function can be expressed as a linear combination of the gradient of the constraints:

$$\begin{bmatrix} L_x^T & L_u^T \end{bmatrix} = -\lambda^T \begin{bmatrix} f_x & f_u \end{bmatrix}.$$

Equations (2.6) may also be obtained by constructing the *Hamilton function* H defined as follows

$$H(x, u, \lambda) = L(x, u) + \lambda^T f(x, u) \tag{2.7}$$

and computing a stationary point of H by setting its partial derivatives equal to zero:

$$H_x = L_x + \lambda^T f_x = 0 \tag{2.8}$$

$$H_u = L_u + \lambda^T f_u = 0 \tag{2.9}$$

$$H_\lambda = f(x, u) = 0. \tag{2.10}$$

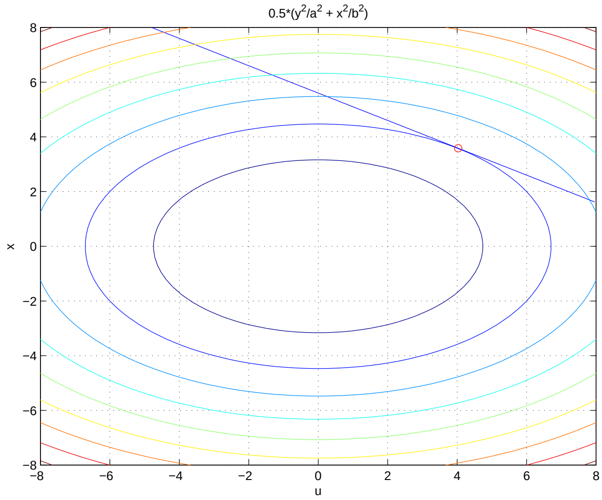

Figure 2.3: Contour plot of $L(x,u)$ with constraint $f(x,u)=0$. The circle denotes the minimum.

Example 2.3: Solve the following optimization problem:

$$\begin{aligned} \min \quad L(x,u) &= \frac{1}{2}\left(\frac{x^2}{a^2}+\frac{u^2}{b^2}\right) \\ \text{s.t.} \quad & \\ f(x,u) &= x+mu-c=0 \end{aligned}$$

Hamilton function:

$$H=\frac{1}{2}\left(\frac{x^2}{a^2}+\frac{u^2}{b^2}\right)+\lambda\left(x+mu-c\right).$$

Stationarity conditions:

$$\begin{aligned} H_x &= \frac{x}{a^2}+\lambda=0 \\ H_u &= \frac{u}{b^2}+\lambda=0 \\ H_\lambda &= x+mu-c=0. \end{aligned}$$

After some short algebraic computations, one obtains:

$$x=a^2c/\Delta, \quad u=b^2mc/\Delta, \quad \lambda=c/\Delta \quad \text{where } \Delta=a^2+m^2b^2.$$

Contour lines of $L(x,u)$ and the constraint $f(x,u)=0$ are illustrated in Fig. 2.3. □

Second Order Conditions. To derive sufficient conditions for a minimum, consider a local second order Taylor expansion of dL and df at a critical point:

$$\begin{aligned} dL &= [L_x^T \; L_u^T]\begin{bmatrix} dx \\ du \end{bmatrix}+\frac{1}{2}\begin{bmatrix} dx \\ du \end{bmatrix}^T\begin{bmatrix} L_{xx} & L_{xu} \\ L_{ux} & L_{uu} \end{bmatrix}\begin{bmatrix} dx \\ du \end{bmatrix}, \\ df &= [f_x^T \; f_u^T]\begin{bmatrix} dx \\ du \end{bmatrix}+\frac{1}{2}\begin{bmatrix} dx \\ du \end{bmatrix}^T\begin{bmatrix} f_{xx} & f_{xu} \\ f_{ux} & f_{uu} \end{bmatrix}\begin{bmatrix} dx \\ du \end{bmatrix}. \end{aligned}$$

Next, the expression $dL+\lambda^T df$ is computed using equations (2.8) and (2.9):

$$dL+\lambda^T\underbrace{df}_{0}=\underbrace{[H_x^T \; H_u^T]}_{0}\begin{bmatrix} dx \\ du \end{bmatrix}+\frac{1}{2}\begin{bmatrix} dx \\ du \end{bmatrix}^T\begin{bmatrix} H_{xx} & H_{ux} \\ H_{xu} & H_{uu} \end{bmatrix}\begin{bmatrix} dx \\ du \end{bmatrix}.$$

Inserting compatibility conditions (2.3) eliminates the dependent variations dx:

$$dL = \frac{1}{2} du^T \begin{bmatrix} -f_x^{-1} f_u \\ I \end{bmatrix}^T \begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{bmatrix} \begin{bmatrix} -f_x^{-1} f_u \\ I \end{bmatrix} du.$$

Since a local minimum requires that $dL \geq 0$ for all arbitrary variations du, the second order conditions are given by:

$$\begin{bmatrix} -f_x^{-1} f_u \\ I \end{bmatrix}^T \begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{bmatrix} \begin{bmatrix} -f_x^{-1} f_u \\ I \end{bmatrix} \succ 0. \tag{2.11}$$

3 Discrete Time Optimal Control

In this section we will derive the necessary equations for computing the optimal control law for a discrete-time system.

3.1 Solution of the Generic Time-Discrete Control Problem

The equations of the discrete time system are given by

$$x_{k+1} = f^k(x_k, u_k), \quad k = i, \dots, N-1, \tag{3.1}$$

where i denotes the start time and N the end of the optimal control problem. Our aim is to find the control law that minimizes a certain *cost function* or *performance index* which is defined as follows:

$$J = \Phi(N, x_N) + \sum_{k=i}^{N-1} L^k(x_k, u_k). \tag{3.2}$$

In a nutshell, the optimization problem is given by

$$\begin{array}{l} \min J \\ \text{s.t.} \\ x_{k+1} = f^k(x_k, u_k), \quad k = i, \dots, N-1. \end{array} .$$

Examples for cost functions:

- Minimum time: $L^k = 0$, $\Phi = N$ or $L^k = 1$, $\Phi = 0$.
- Minimum fuel: $L^k = |u_k|$, $\Phi = 0$.
- Minimum energy: $L^k = |u_k|^2$, $\Phi = 0$.

Similar to the static optimization with equality constraints introduced in Section 2.2, the system constraints (3.1) are taken into account by appending them to the cost function via Lagrange multipliers:

$$\bar{J} = \Phi(N, x_N) + \sum_{k=i}^{N-1} \left[L^k(x_k, u_k) + \lambda_{k+1}^T (f^k(x_k, u_k) - x_{k+1}) \right]. \tag{3.3}$$

With the Hamilton function H^k defined as

$$H^k(x_k, u_k) = L^k(x_k, u_k) + \lambda_{k+1}^T f^k(x_k, u_k),$$

equation (3.3) can be rewritten as follows:

$$\bar{J} = \Phi(N, x_N) - \lambda_N^T x_N + H^i(x_i, u_i) + \sum_{k=i+1}^{N-1} \left[H^k(x_k, u_k) + \lambda_k^T x_k \right]. \tag{3.4}$$

To derive necessary conditions for a minimum of $\bar{J}$, its variation $d\bar{J}$ has to be considered:

$$\begin{aligned} d\bar{J} = {} & (\Phi_{x_N} - \lambda_N)^T dx_N + (H_{x_i}^i)^T dx_i + \sum_{k=i+1}^{N-1} (H_{x_k}^k - \lambda_k)^T dx_k + (H_{u_i}^i)^T du_i + \\ & + \sum_{k=i+1}^{N-1} (H_{u_k}^k)^T du_k + \sum_{k=i+1}^{N} (H_{\lambda_k}^{k-1} - x_k)^T d\lambda_k. \end{aligned} \tag{3.5}$$

For a minimum, the increment $d\bar{J}$ must be ≥ 0 for all variations dx_k, du_k, $d\lambda_k$. Thus all factors of (3.5) are required to equal zero:

$$\left.\begin{array}{lllll} x_{k+1} & = & \dfrac{\partial H^k}{\partial \lambda_{k+1}} & = & f^k(x_k, u_k) & \text{state equation,} \\ \lambda_k & = & \dfrac{\partial H^k}{\partial x_k} & = & \left(\dfrac{\partial f^k}{\partial x_k}\right)^T \lambda_{k+1} + \dfrac{\partial L^k}{\partial x_k} & \text{costate equation,} \\ 0 & = & \dfrac{\partial H^k}{\partial u_k} & = & \left(\dfrac{\partial f^k}{\partial u_k}\right)^T \lambda_{k+1} + \dfrac{\partial L^k}{\partial u_k} & \text{stationarity equation.} \end{array}\right\} \tag{3.6}$$

Note that in contrast to the state equation, the costate equation runs backward in time, i.e. future λs at time $k+1$ yield present ones at time k. Last but not least, also the boundary conditions follow from $d\bar{J} \geq 0$:

$$\left(\frac{\partial \Phi}{\partial x_N} - \lambda_N\right)^T dx_N = 0 \tag{3.7}$$

and

$$\left(\frac{\partial H^i}{\partial x_i}\right)^T dx_i = 0. \tag{3.8}$$

Exemplary boundary conditions:

- Initial state given: $x_i = x_0 \quad \Longrightarrow \quad dx_i = 0$.
- Final state free: arbitrary variations $dx_N \quad \Longrightarrow \quad \Phi_{x_N} - \lambda_N = 0$.

Example 3.1: Compute the optimal control law that minimizes

$$J = \frac{r}{2} \sum_{k=0}^{N-1} u_k^2$$

subject to the discrete time system

$$x_{k+1} = a x_k + b u_k$$

for fixed initial state x_0 and fixed final state x_N.

For this example, equations (3.6) become

$$\left.\begin{array}{lll} x_{k+1} & = & a x_k + b u_k, \\ \lambda_k & = & a \lambda_{k+1}, \\ 0 & = & b \lambda_{k+1} + r u_k. \end{array}\right\} \tag{3.9}$$

Solving the second equation of (3.9) yields immediately an equation for λ_k:

$$\lambda_k = a^{N-k} \lambda_N. \tag{3.10}$$

The control law u_k can be computed explicitly from the last equation:

$$u_k = -\frac{b}{r} \lambda_{k+1}. \tag{3.11}$$

Substituting both into the first equation of (3.9) yields a difference equation for x_{k+1}:

$$x_{k+1} = a x_k - \gamma a^{N-k-1} \lambda_N,$$

where $\gamma = \frac{b^2}{r}$. This equation can be solved analytically:

$$x_k = a^k x_0 - \lambda_N a^{N-k} \gamma \frac{1-a^{2k}}{1-a^2}. \tag{3.12}$$

To compute the yet unknown λ_N, the boundary condition for the final state being x_N has to be considered, i.e., equation (3.12) must conform to the boundary condition:

$$x_N = a^N x_0 - \Lambda \lambda_N,$$

where $\Lambda = \gamma \frac{1-a^{2N}}{1-a^2}$. Thus,

$$\lambda_N = \frac{1}{\Lambda}(a^N x_0 - x_N)$$

Finally, plugging this and (3.10) in (3.11) yields the control law:

$$u_k^o = \frac{b}{r\Lambda}(x_N - a^N x_0) a^{N-k-1}.$$

Note that if the final state is equal to $a^N x_0$, i.e., the state to which the system would go without doing anything, the control law becomes zero. Since every control action u gets punished in the cost function, this is indeed a wise decision. Just for reasons of completeness we state the cost function without derivation:

$$J^o = \frac{1}{2\Lambda}(x_N - a^N x_0)^2.$$

□

3.2 Linear Quadratic (LQ) Regulator

The equations derived in the previous section apply to the generic case of nonlinear systems and arbitrary cost functions. Unfortunately, explicit solutions for the control law can be obtained only for some special cases such as linear systems with quadratic performance index. This so-called *LQ Regulator Problem* will be considered in the following. Surprisingly, we will be able to obtain a feedback control law in case of a free final state.

3.2.1 LQ Problem with Finite Time Horizon

The state equations of the linear system are given by:

$$x_{k+1} = A_k x_k + B_k u_k, \tag{3.13}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and A_k, B_k are appropriate matrices. The cost function J for the problem starting at time i and ending at time N is defined as:

$$J = \frac{1}{2} x_N^T S_N x_N + \frac{1}{2} \sum_{k=i}^{N-1} (x_k^T Q_k x_k + u_k^T R_k u_k), \tag{3.14}$$

where S_N and Q_k are symmetric positive semidefinite matrices and R_k is a symmetric, positive definite matrix. For this optimal control problem, state-, costate-, and stationarity-equation (3.6) become:

$$\begin{aligned}
x_{k+1} &= A_k x_k + B_k u_k, && (3.15)\\
\lambda_k &= Q_k x_k + A_k^T \lambda_{k+1}, && (3.16)\\
0 &= R_k u_k + B_k^T \lambda_{k+1}. && (3.17)
\end{aligned}$$

By computing u_k from (3.17),

$$u_k = -R_k^{-1} B_k^T \lambda_{k+1}, \tag{3.18}$$

equations (3.15) and (3.16) can be rewritten as:

$$\begin{bmatrix} x_{k+1} \\ \lambda_k \end{bmatrix} = \begin{bmatrix} A_k & -B_k R_k^{-1} B_k^T \\ Q_k & A_k^T \end{bmatrix} \begin{bmatrix} x_k \\ \lambda_{k+1} \end{bmatrix}.$$

For a free final state x_N, we obtain a boundary condition for λ_N according to equation (3.7):

$$S_N x_N = \lambda_N.$$

In order to compute the optimal control law, a little trick has to be applied: assume that there exists a S_k such that

$$S_k x_k = \lambda_k. \tag{3.19}$$

Thus the state-equation (3.15) becomes

$$x_{k+1} = A_k x_k - B_k R_k^{-1} B_k^T S_{k+1} x_{k+1}.$$

and is solved for x_{k+1}:

$$x_{k+1} = (I + B_k R_k^{-1} B_k^T S_{k+1})^{-1} A_k x_k. \tag{3.20}$$

Together with definition (3.19) the costate equation (3.15) can be written as:

$$S_k x_k = Q_k x_k + A_k^T S_{k+1} x_{k+1}.$$

Plugging equation (3.20) into the previous one yields:

$$S_k x_k = \left[Q_k + A_k^T S_{k+1} \left(I + B_k R_k^{-1} B_k^T S_{k+1} \right)^{-1} A_k \right] x_k. \tag{3.21}$$

Since above equation must hold for arbitrary states x_k, we require S_k to equal the square bracket term:

$$S_k = Q_k + A_k^T S_{k+1} \left(I + B_k R_k^{-1} B_k^T S_{k+1} \right)^{-1} A_k. \tag{3.22}$$

This is the famous *Riccati Recursion* which enables us to recursively precompute the matrices S_k starting at S_N which is given by the performance index (3.14). Note that since S_N is symmetric and positive definite, all S_k will have these properties too.

Finally, the control law is obtained by inserting definition (3.19) and state equation (3.15) in (3.18) :

$$u_k = -R_k^{-1} B_k^T S_{k+1} x_{k+1} = -R_k^{-1} B_k^T S_{k+1} (A_k x_k + B_k u_k)$$

Solving for u_k yields an expression for the optimal control law:

$$u_k^o = \underbrace{- \left(B_k^T S_{k+1} B_k + R_k \right)^{-1} B_k^T S_{k+1} A_k}_{K} x_k = -K x_k. \tag{3.23}$$

It is important to note that we managed to express u_k^o as a linear function of the current state x_k, i.e., we found a closed loop control for the LQ regulator problem.

For completeness, we state the cost function with respect to the optimal control law (without derivation):

$$J^o(x_i) = \frac{1}{2} x_i^T S_i x_i.$$

3.2.2 Time Invariant LQ Problem with Infinite Time Horizon

In the time invariant case A_k, B_k, R_k and Q_k aren't dependent on time k. To obtain the control law in case of an infinite time horizon (or equivalently $k \to -\infty$), we need to compute the limit $S_\infty = \lim_{k\to-\infty} S_k$. Using the condition $S_k \equiv S_{k+1} \equiv S_\infty$ in (3.22) yields the *discrete algebraic Riccati equation* (DARE):

$$S_\infty = Q + A^T S_\infty \left(I + BR^{-1}B^T S_\infty\right)^{-1} A. \tag{3.24}$$

What is left is to answer the question when S_∞ is bounded. In the following we will state conditions to guarantee a bounded solution:

Stabilizability The system (A, B) is stabilizable if $\exists K$ such that $A - BK$ is stable, i.e. for discrete time systems $|\lambda_i(A - BK)| < 1$.

Theorem 3.1 (without proof) *If (A, B) is stabilizable, then $S_\infty = \lim_{k\to-\infty} S_k$ is bounded and a positive definite solution of the DARE* (3.24)*.*

Since $Q \succeq 0$ there exists a matrix Q such that $Q = C^T C$. If (A, C) is observable, the costs J will indicate if x goes to infinity: $\sum x_k^T Q x_k = \sum ||Cx_k||^2 = \sum ||y_k||^2$.

Theorem 3.2 (without proof) *Let $Q = C^T C \succeq 0$. If (A, C) is observable and (A,B) is stabilizable, S_∞ is the unique positive definite solution of the DARE* (3.24) *and the closed loop plant $(A - BK_\infty)$ is asymptotically stable.*

Example 3.2: Consider the following 1-dimensional optimal control problem with infinite time horizon:

$$\begin{aligned}
&\min J = \frac{1}{2}\sum_k \left(qx_k^2 + ru_k^2\right) \\
&\text{s.t.} \\
&x_{k+1} = ax_k + bu_k, \qquad |a| < 1 \text{ (stable system).}
\end{aligned}$$

and compute the optimal controller gain K_∞ in case of very expensive control, i.e., $r \to \infty$.

For this example, the DARE (3.24) simplifies to:

$$s_\infty = a^2\left(s_\infty - \frac{s_\infty^2 b^2}{r + b^2 s_\infty}\right) + q,$$

and respectively k_∞ is given by:

$$k_\infty = \frac{abs_\infty}{r + b^2 s_\infty}.$$

Letting $r \to \infty$ yields the following limit for s_∞ and k_∞:

$$s_\infty = \frac{q}{1 - a^2} \quad \text{and} \quad k_\infty = 0.$$

This is a reasonable result since we required our system to be stable, i.e., its state goes to 0 without any controller action, and control is very expensive. □

4 Continuous Time Optimal Control

Knowing how to compute the optimal control law for discrete time systems from Section 3, we move on to optimal control of continuous time systems.

4.1 Solution of the Generic Continuous Time Optimal Control Problem

Regard the continuous time system

$$\dot{x} = f(x, u, t). \tag{4.1}$$

Similar to Section 3 we introduce a cost functional $J(x, u, t)$ as a performance measure for our problem starting at time t_o and ending at time T:

$$J(x, u, t) = \Phi(x(T), T) + \int_{t_0}^{T} L(x(t), u(t), t)dt. \tag{4.2}$$

Our aim is again to find the control law $u(t)$ that minimizes the costs J and furthermore satisfies additional constraints at final time T:

$$\Psi(x(T), T) = 0. \tag{4.3}$$

To sum up, the optimal control problem is given by

$$\begin{aligned}
&\min J(x, u, t)\\
&\text{s.t.}\\
&\dot{x} = f(x, u, t)\\
&\Psi(x(T), T) = 0.
\end{aligned}$$

To get rid of the constraints, the Lagrange multiplier method is applied again yielding the extended performance index $\bar{J}$:

$$\bar{J} = \Phi(x(T), T) + \nu^T \Psi(x(T), T) + \int_{t_0}^{T} \left[L(x(t), u(t), t) + \lambda^T(t)(f(x, u, t) - \dot{x})\right] dt, \tag{4.4}$$

where ν and $\lambda(t)$ denote the Lagrange multipliers corresponding to final state and system dynamic constraints.

Next, we define the Hamiltonian

$$H(x, u, t) = L(x, u, t) + \lambda^T f(x, u, t). \tag{4.5}$$

and compute the variation $d\bar{J}$ to derive necessary conditions for a minimum. After some short algebraic computations and using integration by parts, one obtains:

$$\begin{aligned}
d\bar{J} =& (\Phi_x + \Psi_x^T \nu + \lambda)^T dx(T) + (\Phi_t + \Psi_t \nu + H - \lambda^T \dot{x} + \lambda^T \dot{x})dt(T) + \Psi^T(T)d\nu + \lambda^T dx(t_0) - \\
&(H - \lambda^T \dot{x} + \lambda^T \dot{x})dt(t_0) + \int_{t_0}^{T} \left[(H_x + \dot{\lambda})^T dx + (H_\lambda - \dot{x})^T d\lambda + H_u^T du\right] dt.
\end{aligned} \tag{4.6}$$

For a minimum $d\bar{J}$ must be ≥ 0 for arbitrary variations, i.e., the bracket terms which are multiplied with an increment must vanish. Thus, analyzing the terms in the integral yields state, costate and stationarity equation:

$$\left.\begin{array}{rcll}
\dot{x} & = & \dfrac{\partial H}{\partial \lambda} & \text{state equation,}\\
\dot{\lambda} & = & -\dfrac{\partial H}{\partial x} & \text{costate equation,}\\
0 & = & \dfrac{\partial H}{\partial u} & \text{stationarity equation.}
\end{array}\right\} \tag{4.7}$$

Furthermore, boundary conditions for free final state and/or time are obtained from the first two terms in (4.6):

$$\left(\Phi_x + \Psi_x^T \nu - \lambda\right)^T dx(T) + (\Phi_t + \Psi_t \nu + H)_T \, dT = 0. \tag{4.8}$$

Properties of the Hamiltonian Consider the absolute time derivative of the Hamiltonian together with equations (4.7):

$$\dot{H} = \underbrace{\left(\frac{\partial H}{\partial x}\right)^T}_{-\dot{\lambda}^T} \dot{x} + \underbrace{\left(\frac{\partial H}{\partial u}\right)^T}_{0} \dot{u} + \underbrace{\left(\frac{\partial H}{\partial \lambda}\right)^T}_{\dot{x}} \dot{\lambda} + \frac{\partial H}{\partial t} = -\dot{\lambda}^T x + \dot{x}^T \lambda + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

If $f = f(x,u)$ and $L = L(x,u)$ are independent of time t, the time derivative of $H = H(x,u)$ is equal to zero and and as a consequence H must be constant along trajectories of the system[2].

Example 4.1: Lagrange's Equation of Motion. Find the function $x(t)$ that minimizes the following functional:

$$J = \int_0^T L(x, \dot{x}) dt$$

by using the "fake" dynamics $\dot{x} = u$ above problem can be cast into the standard form:

$$\begin{aligned} &\min J = \min \int_0^T L(x,u)dt \\ &\text{s.t.} \\ &\dot{x} = u. \end{aligned}$$

Thus, the Hamiltonian is

$$H = L(x,u) + \lambda u$$

and state, costate, and stationarity equations (4.7) are given by

$$\dot{x} = u \tag{4.9}$$
$$\dot{\lambda} = -L_x \tag{4.10}$$
$$0 = L_u + \lambda \tag{4.11}$$

Inserting the time derivative of (4.11) together with (4.9) in (4.10) yields the well-known *La-*

[2]In physics a constant Hamiltonian is interpreted as conservation of energy.

grange equation of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0.$$

□

Example 4.2: What is the shortest path between two points? See Fig. 4.1.

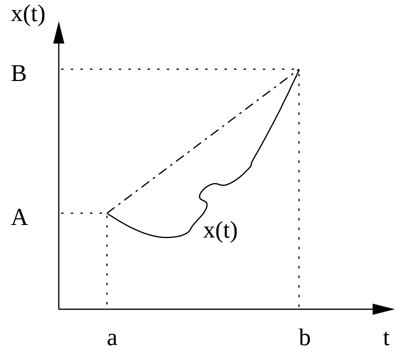

Figure 4.1: Shortest path between two points.

$$J = \int_a^b \sqrt{1 + \dot{x}^2} dt$$

Substituting $\dot{x}$ by u transforms the problem into an optimal control problem of standard form:

$$\begin{aligned} &\min \int_a^b \sqrt{1 + u^2} dt \\ &\text{s.t.} \\ &\dot{x} = u. \end{aligned}$$

Hence, $L = \sqrt{1 + u^2}$ and equations (4.9)–(4.11) become:

$$\dot{x} = u \tag{4.12}$$

$$\dot{\lambda} = 0 \tag{4.13}$$

$$0 = \frac{1}{2}(1 + u^2)^{-\frac{1}{2}} 2u + \lambda \tag{4.14}$$

Equation (4.13) implies that λ must be a constant and due to (4.14) u must be constant too. As a consequence, $x(t)$ must be a linear function: $x(t) = cx + d$. Inserting the boundary conditions yields a straight line connecting points A and B. Note that the boundary conditions weren't necessary to obtain the structure of the optimal solution – it's already contained in the state, costate, and stationarity equation. □

4.2 Linear Systems

Consider the linear system

$$\dot{x} = Ax + Bu \tag{4.15}$$

with respect to the given value function

$$J = \frac{1}{2} x(T)^T S_T x(T) + \frac{1}{2} \int_0^T [x^T Q x + u^T R u] dt. \tag{4.16}$$

Hence, the Hamiltonian is given by

$$H = \frac{1}{2}(x^T Q x + u^T R u) + \lambda^T (Ax + Bu)$$

and the state, costate, and stationarity equations corresponding to our optimal control problem are as follows:

$$\begin{aligned}
\dot{x} &= Ax + Bu, && (4.17)\\
\dot{\lambda} &= -Qx - A^T\lambda, &&\\
0 &= Ru + B^T\lambda. && (4.18)
\end{aligned}$$

Computing the optimal control law from (4.18)

$$u = -R^{-1}B^T\lambda$$

and inserting it in (4.17) yields a 2-point boundary value problem which must be solved according to the boundary conditions:

$$\begin{aligned}
\dot{x} &= Ax - BR^{-1}B^T\lambda\\
\dot{\lambda} &= -Qx - A^T\lambda.
\end{aligned}$$

Analytical solutions can be obtained in the following cases:

- Fixed final state and $Q = 0$ (see [1, Section 3.3, p. 165]) and
- Free final state, yields closed loop control (see [1, Section 3.3, p. 170]).

4.2.1 Model Reduction - Balanced Realizations

Consider again a linear plant

$$\begin{aligned}
\dot{x} &= Ax + Bu && (4.19)\\
y &= Cx, && (4.20)
\end{aligned}$$

where y denotes the output of the system and A is Hurwitz-stable, i.e., $\mathrm{Re}(\lambda(A)) < 0$. The transfer function for the system above is as follows:

$$G(s) = C(sI - A)^{-1}B.$$

Our aim is to find a new transfer function $G_r(s)$ that has fewer poles than the n ones from $G(s)$ but still describes the behaviour of the system properly, i.e., the error

$$\big|\big|G(s) - G_r(s)\big|\big|_\infty = \sup_\omega \big|G(\omega) - G_r(\omega)\big| = \sup_\omega \bar{\sigma}(G(\omega) - G_r(\omega)),$$

is minimized. In above expression, $\bar{\sigma}$ denotes the maximum singular value.

To find an approximate solution to this problem, we firstly compute the optimal control law that drives the plant from initial state equal to zero to final state equal to x_0 in infinite time:

$$J = \frac{1}{2}\int_0^\infty u^T u dt.$$

We just state the solution of this optimal control problem without derivation (check it!):

$$u^* = B^T e^{A^T t} W_c^{-1} x_0, \tag{4.21}$$

where W_c denotes the *controllability gramian*:

$$W_c = \int_0^\infty e^{At} B B^T e^{A^T t} dt. \tag{4.22}$$

The cost function, i.e., the optimal energy are ellipsoids:

$$J^* = \frac{1}{2} x_0^T W_c^{-1} x_0.$$

Next, we consider the energy of the system's output with initial state x_0 and input $u(t)$ set to zero:

$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned}$$

and

$$J = \int_0^\infty \frac{1}{2} y^T y.$$

It is easy to show that the energy of the output is again a quadratic form of x_0:

$$J^* = \frac{1}{2} x_0^T W_o x_0,$$

where W_o denotes the *observability gramian*

$$W_o = \int_0^\infty e^{A^T t} C^T C e^{At} dt. \tag{4.23}$$

Remark In practice the gramians (4.22) and (4.23) aren't computed by doing the integration but rather by solving the corresponding Lyapunov equations:

$$\begin{aligned} AW_c + W_c A^T + BB^T &= 0 \\ A^T W_o + W_o A + C^T C &= 0. \end{aligned}$$

Note that in both cases (minimum energy to take system to x_0 and zero input) the energy is a quadratic form. The eigenvectors of W_c and W_o represent the principal axis of the resulting ellipsoids and the eigenvalues their corresponding lengths. The eigenvalues are a measure of how "controllable" respectively "observable" a state is. If we want to find out which states contribute much to the input-output behaviour we have to take into account both controllability and observability. Unfortunately, the ellipsoids generated by W_c and W_o are usually not aligned. Thus we are looking for a coordinate transformation $x = Tz$ which lets the ellipsoids coincide:

$$\begin{aligned} \dot{z} &= \tilde{A} z + \tilde{B} u \\ y &= \tilde{C} z, \end{aligned}$$

where $\tilde{A} = TAT^{-1}$, $\tilde{B} = TB$, $\tilde{C} = CT^{-1}$. Furthermore, $\tilde{W}_c = TW_cT^T$ and $\tilde{W}_o = T^{-T} W_o T^{-1}$. Note that although the transformation changes the eigenvalues of W_c and W_o, the singular values of their product $\sigma_i = \sqrt{\lambda(W_c W_o)}$ are invariant. The following theorem states the existence of a transformation with the properties mentioned above:

Theorem 4.1 *Given two symmetric positive definite matrices W_c, W_o, there exists a transformation T, such that*

$$TW_cT^T = T^{-T} W_o T^{-1} = diag(\sigma_1, \ldots, \sigma_n)$$

After applying T the transfer function of the system is given by

$$G(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} = \left[\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{B}_2 \\ \hline \tilde{C}_1 & \tilde{C}_2 & 0 \end{array}\right],$$

where $\tilde{A}_{11}$ is an $r \times r$ matrix.

Theorem 4.2 *Let the approximating transfer function $G_r(s)$ be defined as follows:*

$$G_r(s) = \left[\begin{array}{c|c} \tilde{A}_{11} & \tilde{B}_1 \\ \hline \tilde{C}_1 & 0 \end{array}\right].$$

Then following inequality holds:

$$||G(s) - G_r(s)||_\infty \leq 2\sum_{r+1}^{n} \sigma_i.$$

5 Dynamic Programming

Principle: If we know an optimal path, then every sub-path is optimal too.

5.1 Discrete Time Dynamic Programming

To illustrate the basic idea of dynamic programming for discrete systems, we start off with an example:

Example 5.1: Find the path with the lowest costs given by Fig. 5.1 for going from A to B.

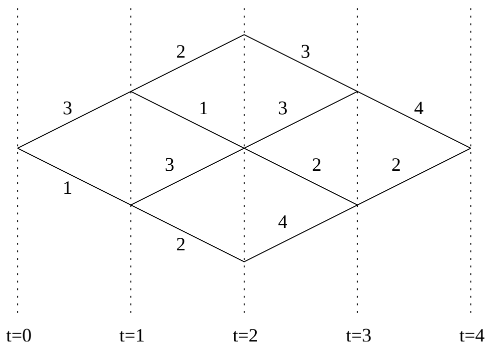

Figure 5.1: Find the cheapest path to get from A to B.

Solution procedure:

- Start going backwards in time.
- Replace future decisions by the current value function.
- In order to make a local decision at time k minimize the sum of the current decision at time k and the value function at time $k+1$. This gives the value function at time k.

Applying above rules yields value function and control law (in which direction one should go) step by step. Fig. 5.2 shows the results for each of the different time steps. □

We will now carry over the ideas from this introductory example to solve general discrete time problems. Regard again the generic state equations given by

$$x_{k+1} = f_k(x_k, u_k)$$

with the corresponding cost functional

$$J(x_i) = \Phi(x_N, N) + \sum_{k=i}^{N-1} L_k(x_k, u_k).$$

To compute the optimal decision u_k at current time k we assume that the optimal costs and control law for all future time steps are already known. The optimal costs J_k are given by the sum of the costs of the current decision and the optimal costs for the subsequent time steps:

$$J_k^*(x_k) = \min_{u_k} \left[L_k(u_k, x_k) + J_{k+1}^*(x_{k+1}) \right].$$

Solving this equation backwards in time for all states yields the optimal control law.

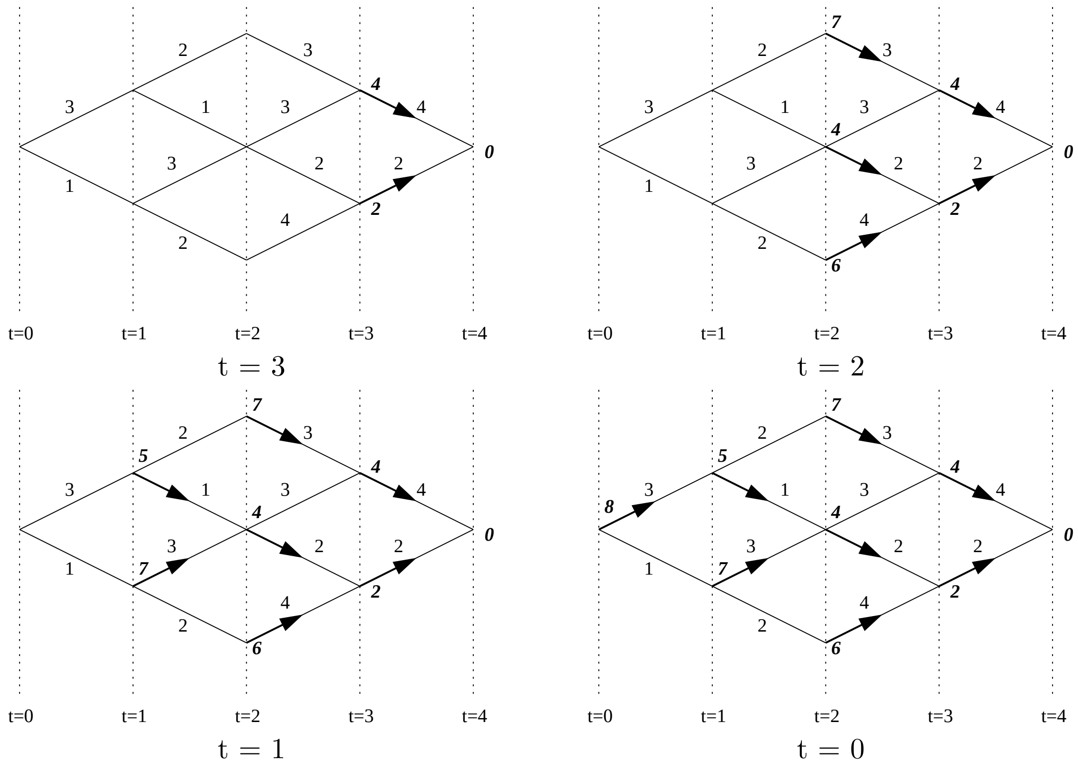

Figure 5.2: Solution procedure for the optimal path problem

5.2 Continuous Time Dynamic Programming

In this section we will consider dynamic programming for continuous time systems of the standard form

$$\dot{x} = f(x, u)$$

with the already well–known cost functional

$$J(x, u, t) = \Phi(x(T), T) + \int_t^T L(x, u, t)dt. \tag{5.1}$$

5.2.1 Derivation of the Hamilton-Jacobi-Bellman (HJB) Equation

In the following we will derive the HJB equation as a necessary condition for the value function. Before we proceed, we split the integral into two parts:

$$J(x, u, t) = \Phi(x(T), T) + \int_t^{t+\Delta t} L(x, u, t)dt + \int_{t+\Delta t}^T L(x, u, t)dt$$

Analog to the solution procedure applied in the previous section, we assume that the optimal solution for the second part is already known and the optimal costs are thus given by

$$J^*(x, t) = \min_{\substack{u(\tau) \\ t \le \tau \le t+\Delta t}} \left[\int_t^{t+\Delta t} L(x, u, t)dt + J^*(x, t + \Delta t) \right].$$

$J^*(x, t + \Delta t)$ is now replaced by a first order Taylor approximation:

$$J^*(x, t) = \min_u \left[\int_t^{t+\Delta t} L(x, u, t)dt + J^*(x, t) + J_t^*(x, t)\Delta t + J_x^*(x, t)\Delta x \right].$$

Dividing by Δt and letting Δt tend to zero yields the *Hamilton-Jacobi-Bellman* equation

$$-J_t^*(x,t) = \min_u \left[L(x,u,t) + J_x^{*T}(x,t) f(x,u,t)\right]. \tag{5.2}$$

The terminal condition for this scaler first order PDE follows directly from the fact that the value function must equal the end cost term of (5.1) at final time T:

$$J(x(T),T) = \Phi(x(T),T).$$

By defining the Hamiltonian $H(x,u,\lambda,t) = L(x,u,t) + \lambda^T f(x,u,t)$ (5.2) can be rewritten as

$$-\frac{\partial J^*(x,t)}{\partial t} = \min_u H\left(x, u, \frac{\partial J^*(x,t)}{\partial x}, t\right).$$

5.2.2 The HJB Inequality

Note that equality in (5.2) holds only for the optimal control law that minimizes the right hand side. Consequently, following inequality must hold for all other u:

$$-J_t^*(x,t) \le L(x,u,t) + J_x^{*T}(t,x) f(x,u,t)$$

Theorem 5.1 *Let V be a function such that*

$$-V_t(x,t) \le L(x,u,t) + V_x^T(x,t) f(x,u,t) \tag{5.3}$$

with boundary condition

$$V(x(T),T) = \Phi(x(T),T). \tag{5.4}$$

Then

$$V(x,t) \le J^*(t,x). \tag{5.5}$$

Proof:

$$\begin{aligned} J^*(x,t) &= \Phi(x(T),T) + \int_t^T L(x,u,t)dt \ge \Phi(x(T),T) + \int_t^T \left[-V_t(x,t) - V_x^T(x,t) f(x,u,t)\right] dt = \\ &= \Phi(x(T),T) + \int_t^T -\frac{d}{dt}\left(V(x(t),t)\right) dt = \Phi(x(T),T) + V(x,t) - \underbrace{V(x(T),T)}_{\Phi(x(T),T)} = \\ &= V(x,t). \end{aligned}$$

■

Remark: If V_1 and V_2 are solutions of (5.3)-(5.4), then every convex combination of V_1 and V_2 is a solution too, i.e., $\lambda V_1 + (1-\lambda) V_2$ with $\lambda \in [0,1]$.

5.2.3 LQ Regulator - Once Again

Consider again the linear system (4.15) with cost functional (4.16) from Section 4.2, page 16. In the following we will derive the optimal control law for the problem with free final state by using the HJB equation. To compute the control law that maximizes the Hamiltonian, we simply set its partial derivative equal to zero:

$$\frac{\partial H}{\partial u} = Ru + B^T \lambda = 0.$$

Hence

$$u = -R^{-1} B^T \lambda.$$

Inserting this into the HJB equation yields the following PDE

$$
\begin{aligned}
-J_t^* &= \frac{1}{2}x^TQx + \frac{1}{2}(-R^{-1}B^TJ_x^*)^TR(-R^{-1}B^TJ_x^*) + J_x^{*T}(Ax - BR^{-1}B^TJ_x^*) \\
-J_t^* &= \frac{1}{2}x^TQx - \frac{1}{2}J_x^{*T}BR^{-1}B^TJ_x^* + J_x^{*T}Ax
\end{aligned}
$$

with terminal condition

$$J^*(x(T), T) = \frac{1}{2}x^TS_Tx.$$

By using the ansatz

$$J^*(x,t) = \frac{1}{2}x^TS(t)x,$$

we obtain

$$-\frac{1}{2}x^T\dot{S}x = \frac{1}{2}x^TQx - \frac{1}{2}x^TSBR^{-1}B^TSx + x^TSAx.$$

Note that all terms in above differential equation are quadratic in the state. Since it must hold for all x, we obtain the following *Matrix Riccati Differential Equation* as a solution:

$$
\begin{aligned}
-\dot{S} &= Q - SBR^{-1}B^TS + SA + A^TS \\
S(T) &= S_T.
\end{aligned}
\tag{5.6}
$$

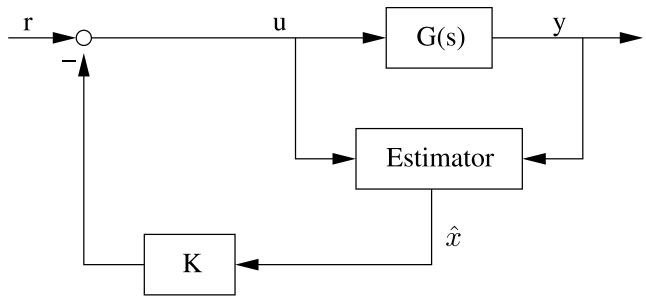

Figure 6.1: Kalman filter in the loop

6 LQG Regulator - Kalman Filter

In Section 5.2.3 we derived the optimal feedback control law for a linear system under the tacit assumption that all states are measurable. However, in reality we often don't have access to the full state space and must get along with the information from the measured output y.

6.1 Kalman Filter

Consider the following state space system with output y:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx. \end{aligned} \tag{6.1}$$

In order to employ the feedback law of the form $u = -Kx$, we must somehow estimate the state x by the measured output y. Figure 6.1 shows the block diagram of the resulting output feedback control law. Being a stochastic approach, the *Kalman Filter* tries to compute the best estimate $\hat{x}$ for the system under the assumption of additional process and measurement noise:

$$\begin{aligned} \dot{x} &= Ax + Bu + Gw \\ y &= Cx + v, \end{aligned} \tag{6.2}$$

where w and v denote white noise given by

$$\begin{aligned} E[w(t)w(t+\tau)^T] &= \delta(\tau)Q, \quad Q \succeq 0 \\ E[v(t)v(t+\tau)^T] &= \delta(\tau)R, \quad R \succ 0. \end{aligned}$$

For computing the estimated state $\hat{x}$ we use

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$$

or, with the estimated output $\hat{y} = C\hat{x}$,

$$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly. \tag{6.3}$$

The dynamics of the estimation error $e = x - \hat{x}$ are thus given by

$$\dot{e} = (A - LC)e - Gw - Lv. \tag{6.4}$$

Note that if $(A - LC)$ is Hurwitz, the error would tend to zero in case of absence of noise. Our aim is now to find the best possible estimator gain L that makes the error small or, equivalently, minimizes the trace of its covariance

$$P(t) = E[e(t)e^T(t)].$$

It turns out that the error covariance $P(t)$ must satisfy the algebraic Riccati equation

$$AP + PA^T + GQG^T - PC^TR^{-1}CP = 0 \tag{6.5}$$

and the Kalman gain L is given by (for a detailed derivation refer to [1, pp. 459–470])

$$L = PC^TR^{-1}. \tag{6.6}$$

Let us rewrite the differential equations for the state and the error in matrix form:

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \underbrace{\begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix}}_{M} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r + \begin{bmatrix} G \\ G \end{bmatrix} w + \begin{bmatrix} 0 \\ L \end{bmatrix} v,$$

$$y = \begin{bmatrix} c & 0 \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}.$$

Separation principle for stability. If both $A - LC$ and $A - BK$ are stable, i.e., stable controller and stable estimator, then the full control interconnection (controller + estimator) is stable.

6.2 Stability Margins for the LQ/LQG Regulator

In the following section we investigate how a perturbation δ of the feedback gain K, i.e. $u = -\delta Kx$, can affect stability of the system.

Case 1: State Space Feedback Control Consider again the LQ problem with full measurable state space from section 5.2.3. In case of an infinite time horizon (5.6) becomes

$$SA + A^TS = -Q + SBR^{-1}B^TS. \tag{6.7}$$

and the feedback gain is given by

$$K = R^{-1}B^TS. \tag{6.8}$$

Subtracting $(\delta BK)^TS$ and $S\delta BK$ from (6.7) and using (6.8) on the right hand side yields:

$$S(A - \delta BK) + (A - \delta BK)^TS = -Q - (2\delta - 1)[SBR^{-1}B^TS]. \tag{6.9}$$

This can be considered as a Lyapunov equation for the system $(A-\delta BK)$ and since $SBR^{-1}B^TS$ is positive definite, the right side of (6.9) is negative definite for $\delta > \frac{1}{2}$ and hence $(A - \delta BK)$ is guaranteed to be stable for all δ satisfying this condition.

Case 2: Output Feedback Control There are no guaranteed margins for the LQG regulator. See [2]

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[1] F. L. Lewis and V. L. Syrmos, *Optimal Control*. John Wiley & Sons, Inc., 1995.

[2] J. C. Doyle, “Guaranteed margins for lqg regulators,” *IEEE Transactions on Automatic Control*, vol. AC-23, no. 4, pp. 756–757, 1978.

[3] G. E. Dullerud and F. Paganini, *A Course in Robust Control Theory: a Convex Approach*. New York: Springer, 2000.