Control Systems 2
Lecture 2: Loopshaping design

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Loopshaping

\[ e = \frac{1}{1+L(s)} r - \frac{1}{1+L(s)} G_d(s)d + \frac{L(s)}{1+L(s)} n \]
Loopshaping

\[
e = \frac{1}{1 + L(s)} r - \frac{1}{1 + L(s)} G_d(s) d + \frac{L(s)}{1 + L(s)} n
\]

Performance requirements can be approximated by requirements for \( L(j\omega) \).

\[|L(j\omega)| \gg 1 \implies |S(j\omega)| \ll 1 \quad \text{ (good tracking performance)}\]
\[|L(j\omega_c)| = 1 \quad \text{gives bandwidth} \approx \omega_c\]
\[|L(j\omega)| \ll 1 \implies |T(j\omega)| \ll 1 \quad \text{ (good noise rejection)}\]

Bode gain-phase relationship

For minimum phase stable systems (with \( L(0) > 0 \)),

The phase of \( L(j\omega) \) is determined by the slope of \( |L(j\omega)| \).

If the slope is constant then:

\[
\frac{d|L(j\omega)|}{d\omega} = -20 \text{ dB/decade} \iff \angle L(j\omega) = -90^\circ
\]
\[= -20n \text{ dB/decade} \iff \angle L(j\omega) = -90n^\circ, \quad n = 1, 2, \ldots\]

Slope constraint at crossover:

\[
\frac{d|L(j\omega)|}{d\omega} = -20 \text{ dB/decade} \implies PM \approx 90^\circ
\]

(this requires a constant slope for a wide region of frequency)
Loopshape specifications

1. $|L(j\omega)| \gg 1$ for frequencies requiring high performance.
2. Gain crossover frequency, $\omega_c$, gives closed-loop bandwidth.
3. The slope of $L(j\omega)$ at crossover should be -20dB/decade.
4. The system type is the number of pure integrators in $L(s)$.

Inverting the plant can often be done only approximately.

Inverse-based controller design

Stable, minimum-phase plant

Can choose:

$L(s) = \frac{\omega_c}{s}$

This will give a phase margin of $90^\circ$.

$K(s) = \frac{\omega_c}{s} G^{-1}(s)$

This is often not desirable unless the reference and disturbances affect the output as steps.

Inverting the plant can often be done only approximately.
Example: disturbance process

$$G(s) = \frac{200}{(10s + 1)} \frac{1}{(0.05s + 1)^2}, \quad G_d(s) = \frac{100}{10s + 1}$$

Objectives:

1. Rise time < 0.3 seconds.
2. Overshoot < 5%
3. Disturbance response, $y_d(t)$, satisfies $|y(t)| \leq 1$.
4. Disturbance response, $y_d(t)$, satisfies $|y(t)| < 0.1$ within 3 seconds.
5. $|u(t)| \leq 1$ at all times.

$$|G_d(j\omega)| > 1 \text{ up to } \omega_d \approx 10 \text{ rad/sec} \implies \omega_c \geq 10 \text{ rad/sec.}$$
Example: disturbance process

Inverse-based control design

The plant is stable and minimum-phase:

\[ K_0(s) = \frac{\omega_c}{s} G^{-1}(s) \]

\[ = \frac{\omega_c}{s} \frac{(10s + 1)}{200} (0.05s + 1)^2 \]

\[ \approx \frac{\omega_c}{s} \frac{(10s + 1)}{200} \frac{(0.1s + 1)}{0.01s + 1} \]
Example: inverse-based controller design

The closed-loop bandwidth is very close to $\omega_c \approx 10$ rad/sec.
Example: inverse-based controller design

\[ K_0 = \frac{\omega_c}{s} \frac{(10s + 1)}{200} \frac{(0.1s + 1)}{(0.01s + 1)} \]

Reference tracking

Disturbance response

Loopshaping for disturbance rejection

Disturbance response: \( y_d = S G_d d + \ldots \)

To achieve \(|y_d(t)| \leq 1\) for \(|d(t)| \leq 1\),

we want \(|S G_d(j\omega)| < 1\) for all \(\omega\).

So, we want:

\[ |1 + L(j\omega)| > |G_d(j\omega)| \text{ for all } \omega. \]

or, approximately, \(|L(j\omega)| > |G_d(\omega)| \text{ for all } \omega. \)

Initial guess:

\[ |L_{\min}| \approx |G_d| \text{ or } |K_{\min}| \approx |G^{-1} G_d| \]
Loopshaping for disturbance rejection

Step 1:

Initial guess:

\[ |K_{\text{min}}| \approx |G^{-1}G_d| \]

Choose:

\[ K_0(s) \approx G^{-1}(s)G_d(s) \]

\[ \approx 0.5(0.05s + 1)^2 \]

\[ = 0.5 \]

Example: disturbance rejection design

<table>
<thead>
<tr>
<th>Magnitude</th>
<th>G(jω)</th>
<th>K1(jω)</th>
<th>L1(jω)</th>
<th>S1(jω)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.1</td>
<td>1</td>
<td>10</td>
<td>100</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>1</td>
<td>10</td>
<td>100</td>
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<tr>
<td>1</td>
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<td>100</td>
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<td>10</td>
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<td>1</td>
<td>10</td>
<td>100</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Phase (deg.)</th>
<th>G(jω)</th>
<th>K1(jω)</th>
<th>L1(jω)</th>
<th>S1(jω)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-270</td>
<td></td>
<td>-180</td>
<td>-90</td>
<td>0</td>
</tr>
</tbody>
</table>
Example: disturbance rejection design

\[ K_1 = 0.5 \implies L_1(s) \approx G_d(s) \]

Reference tracking

![Reference tracking graph](image)

Disturbance response

![Disturbance response graph](image)

Loopshaping for disturbance rejection

**Step 2:**

Increase the gain at low frequency.

To get integral action multiply the controller by:

\[ K_2(s) = \frac{s + \omega_I}{s} K_1(s). \]

If \( \omega_I = 0.2 \omega_c \) we get 11° more phase at \( \omega_c \) than with \( K_1 \) alone.

So,

\[ K_2(s) = 0.5 \frac{(s + 2)}{s} \]
Example: disturbance rejection design

\[ K_1 = 0.5 \quad \text{and} \quad K_2 = 0.5 \left( \frac{s+2}{s} \right) \]

Reference tracking

Disturbance response
Loopshaping for disturbance rejection

Step 3:

High frequency correction:

Augment with a lead-lag for “derivative action”.

This will also improve the phase margin.

\[ K_3(s) = K_2(s) \frac{(0.05s + 1)}{(0.005s + 1)} \]

\[ = 0.5 \frac{(s + 2)}{s} \frac{(0.05s + 1)}{(0.005s + 1)} \]

Example: disturbance rejection design
Example: disturbance rejection design

\[ K_1 = 0.5, \quad K_2 = 0.5 \frac{(s + 2)}{s}, \quad K_3 = 0.5 \frac{(s + 2)}{s} \frac{(0.05s + 1)}{(0.005s + 1)} \]

Reference tracking

Disturbance response

Loopshape comparisons

Magnitude

2015-2-22
Example: disturbance rejection design

Loopshape comparisons

\[
\begin{align*}
G(s) + G_d(s) + K(s) & = y_u \\
G(s) & = [r, y_m]
\end{align*}
\]

Effectively choose different transfer functions for \( y_d \) and \( y_r \).

2 degrees-of-freedom designs
2 degrees-of-freedom designs

Reference prefiltering:

\[ G(s) + G_d(s) + K_y(s) + K_r(s) \]

Control structure: \[ u = K_y (K_r r - y_m) \]

So \[ y = \frac{G K_y}{1 + G K_y} K_r r + \frac{1}{1 + G K_y} G_d d = T K_r r + S G_d d \]

with \( L = G K_y \).

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2 degrees-of-freedom designs

Reference prefiltering:

The reference prefiltered step response is \[ y = T K_r r + S G_d d \]

Choose \( K_r = T^{-1} T_{\text{ideal}} \) (approximately)

\[ T(s) \approx \frac{1.5}{0.1s + 1} - \frac{0.5}{0.5s + 1} = \frac{(0.7s + 1)}{(0.1s + 1)(0.5s + 1)} \]

This implies that, \( K_r(s) = \frac{0.5s + 1}{0.7s + 1} \), is a reasonable choice.
2 degrees-of-freedom designs

Reference tracking

\[ K_{3r} = \frac{(0.5s + 1)}{(0.7s + 1)(0.03s + 1)} \]

The additional pole was added to prevent \( u(t) \) peaking above one.

Notes and references

Skogestad & Postlethwaite (2nd Ed.)

Loopshaping: Section 2.6