Internal stability

Definition
A system is internally stable if for all initial conditions, and all bounded signals injected at any place in the system, all states remain bounded for all future time.

\[
\begin{bmatrix}
    y \\
    u
\end{bmatrix} = \begin{bmatrix}
    N_{11}(s) & N_{12}(s) \\
    N_{21}(s) & N_{22}(s)
\end{bmatrix} \begin{bmatrix}
    v \\
    r
\end{bmatrix}
\]

Are all four transfer functions stable?
MIMO concepts: transfer function matrices

\[ y(s) = \begin{bmatrix} y_1(s) \\ \vdots \\ y_{ny}(s) \end{bmatrix} = G(s)u(s) = \begin{bmatrix} G_{11}(s) & \cdots & G_{1n_u}(s) \\ \vdots & \ddots & \vdots \\ G_{ny1}(s) & \cdots & G_{nyu}(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ \vdots \\ u_{nu}(s) \end{bmatrix} \]

\[ G(s) = \begin{bmatrix} G_{11}(s) & \cdots & G_{1n_u}(s) \\ \vdots & \ddots & \vdots \\ G_{ny1}(s) & \cdots & G_{nyu}(s) \end{bmatrix} = \begin{bmatrix} \frac{b_{11}(s)}{a_{11}(s)} & \cdots & \frac{b_{1n_u}(s)}{a_{1n_u}(s)} \\ \vdots & \ddots & \vdots \\ \frac{b_{ny1}(s)}{a_{ny1}(s)} & \cdots & \frac{b_{nyu}(s)}{a_{nyu}(s)} \end{bmatrix} \]

\[ = C(sI - a)^{-1}B + D = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \]

with \( A \in \mathcal{R}^{n \times n}, B \in \mathcal{R}^{n \times n_u}, C \in \mathcal{R}^{ny \times n}, D \in \mathcal{R}^{ny \times n_u}. \)

MIMO block diagrams

Non-commutative

\[ \begin{array}{c}
\xrightarrow{G_1(s)} \\
G_2(s) \\
\xleftarrow{G_1(s)}
\end{array} \quad \neq \quad
\begin{array}{c}
\xrightarrow{G_2(s)} \\
G_1(s) \\
\xleftarrow{G_2(s)}
\end{array} \]

“Push-through” rule

\[ GK(I + GK)^{-1} = G(I + KG)^{-1}K = (I + GK)^{-1}GK \]
MIMO sensitivity and complementary sensitivity functions

\[
y = \left( I + GK \right)^{-1}GK \ r + \left( I + GK \right)^{-1}Gz + \frac{d}{S_o}
\]

\[
u = \left( I + KG \right)^{-1}Kr + \frac{z}{S_i} - \left( I + KG \right)^{-1}Kd
\]

\[
v = \left( I + KG \right)^{-1}Kr - \frac{z}{T_i} - \left( I + KG \right)^{-1}KG \ d
\]

Internal stability

Internally stable \iff \ T(s), G(s)S_o(s) and K(s)S_o(s) stable.

Or, equivalently:

Internally stable \iff \ S_o(s) stable and no RHP cancellations in \ G(s)K(s). (minimal realisations of \ GK & KG contain all RHP poles).
Internal stability

Consequences:

If $G(s)$ has a RHP-zero at $z$ then (if internally stable),

$$
\begin{align*}
L_0(s) &= G(s)K(s) \\
T_0(s) &= G(s)K(s)(I - G(s)K(s))^{-1} \\
S_0(s)G(s) &= (I + G(s)K(s))^{-1}G(s) \\
L_i(s) &= K(s)G(s) \\
T_i(s) &= K(s)G(s)(I + K(s)G(s))^{-1}
\end{align*}
$$

have a RHP-zero at $z$.

Feedback will not move (or remove) the RHP-zero from the closed-loop transfer functions.

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Internal stability

Consequences:

If $G(s)$ has a RHP-pole at $p$ then (if internally stable),

$$
\begin{align*}
L_0(s) &= G(s)K(s) \\
L_i(s) &= K(s)G(s) \\
S_0(s) &= (I + G(s)K(s))^{-1} \\
K(s)S_0(s) &= K(s)(I + G(s)K(s))^{-1} \\
S_i(s) &= (I + K(s)G(s))^{-1}
\end{align*}
$$

have a RHP-pole at $p$.

$$
\begin{align*}
S_0(s) &= (I + G(s)K(s))^{-1} \\
K(s)S_0(s) &= K(s)(I + G(s)K(s))^{-1} \\
S_i(s) &= (I + K(s)G(s))^{-1}
\end{align*}
$$

have a RHP-zero at $p$. 

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Stabilizing controllers

\[
\begin{bmatrix}
  y \\
  u
\end{bmatrix} = \begin{bmatrix} S_0G & T_0 \\ T_i & S_iK \end{bmatrix} \begin{bmatrix} v \\
  r
\end{bmatrix}
\]

Stable plant case:
Define:
\[
Q(s) = K(s)(I + G(s)K(s))^{-1}
\]
Then,
\[
\begin{align*}
S_0G &= (I + GK)^{-1}G = (I - GQ)G \\
T_0 &= GK(I + GK)^{-1} = GQ \\
T_i &= KG(I + KG)^{-1} = QG \\
S_iK &= (I + KG)^{-1}K = Q
\end{align*}
\]
are stable if \( Q \) is stable.

The converse is true:
For every stabilizing controller \( K(s) \),
\[
Q(s) = K(s)(I + G(s)K(s))^{-1}
\]
is also stable.

This is a parameterisation of all stabilizing controllers.

\( Q \)-parameterisation or Youla parametrisation.
**Internal model control (IMC)**

Assume that $G(s)$ is stable and a perfect model: $G(s) = \hat{G}(s)$

\[
y = d + Gu = \frac{GQ}{T_o} r + \frac{(I - GQ)}{S_o} d
\]

\[
u = [(I - QG)^{-1}Q - (I - QG)^{-1}Q][r_y] = [K - K][r_y]
\]

**IMC design (for stable $G(s)$)**

\[Q = K(I + G)K^{-1}, \quad K = (I - QG)^{-1}Q\]

Closed-loop in linear in $Q$:

\[T(s) = G(s)Q(s)\]

Design approach:

\[Q(s) = G(s)^{-1}T_{ideal}(s)\]

or if $G(s) = G_{MP}(s)G_{NMP}(s)$, \[Q(s) = G_{MP}(s)^{-1}T_{ideal}(s)\].

- Relative degree of $T_{ideal}(s) \geq$ relative degree of $G_{MP}(s)$ makes $Q(s)$ proper.
- Cannot invert non-minimum phase parts of $G(s)$.
**IMC design example**

Select a desired closed-loop transfer function:

\[ T_{\text{ideal}}(s) = \frac{\omega_c^2}{(s^2 + \sqrt{2}\omega_c s + \omega_c^2)}, \quad \omega_c = 2.5, \quad S_{\text{ideal}}(s) = 1 - T_{\text{ideal}}(s). \]
**IMC design example**

Invert $G_{MP}(s)$ to get $Q(s)$.

$$Q(s) = G_{MP}(s)^{-1}T_{\text{ideal}}(s) = \frac{(1 + 5s)(1 + s/25)}{5} \frac{\omega_c^2}{(1 + s/5) (s^2 + \sqrt{2}\omega_c s + \omega_c^2)}$$

The actual closed-loop, $T(s)$, is:

$$T(s) = G(s)Q(s) = G_{NMP}(s)T_{\text{ideal}}(s) = \frac{(1 - s/5)}{(1 + s/5) (s^2 + \sqrt{2}\omega_c s + \omega_c^2)}.$$

**Controller:**

$$K(s) = (I - Q(s)G(s))^{-1}Q(s) \quad (5\text{th order controller})$$

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**Magnitude**

- $G(j\omega)$
- $L(j\omega)$
- $K(j\omega)$

**Phase (deg.)**

- $G(j\omega)$
- $L(j\omega)$
- $K(j\omega)$
**IMC design example**

Closed-loop unit step responses:

\[ y_{\text{ideal}}(t) \quad y_{\text{actual}}(t) \]

\[ y_{\text{ideal}}(t) - y_{\text{actual}}(t) \]

**IMC implementation**

\[ G(s) + \hat{G}(s) + Q(s) + ru - d \]

Or ...

\[ G(s) + K(s) \]
MIMO Nyquist stability analysis

For a minimal \( L(s) \),

\[
\begin{align*}
\begin{array}{c}
\text{y} \\
L(s) \\
I
\end{array}
\end{align*}
\]

Closed-loop exponential stability

If and only if,

i) \( \det(I + L(s)) \neq 0 \), for all \( s \in \mathcal{D} \)

ii) The number of CCW encirclements of the origin by \( \det(I + L(s)) \), as \( s \) traverses the boundary of \( \mathcal{D} \), is equal to the number of unstable poles in \( L(s) \).

Small gain theorem

\[
\begin{align*}
\begin{array}{c}
v_1 \\
+ \\
M_1(s) \\
\downarrow y_1 \\
+ \\
M_2(s) \\
\downarrow y_2 \\
v_2
\end{array}
\end{align*}
\]

A sufficient condition for stability

Given \( M_1(s) \) and \( M_2(s) \) stable and minimal with,

\[
\|M_1(s)\| = \gamma_1 \quad \text{and} \quad \|M_2(s)\| = \gamma_2
\]

If \( \gamma_1 \gamma_2 < 1 \) then

then the closed-loop interconnection is stable.

This holds for any induced norm (with the same norm for input and output signals).
The $H_\infty$ norm is a measure of the “size” or “gain” of a system.

If $y(s) = G(s)u(s)$ (and stable) then,

$$\|G(s)\|_{H_\infty} := \sup_{u(s) \neq 0} \frac{\|y(s)\|_2}{\|u(s)\|_2} \quad \text{(induced norm with the space)}$$

$$= \sup_{u(s) \neq 0} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} y(j\omega)^T y(j\omega) d\omega \right)^{1/2}$$

$$= \sup_{u(s) \neq 0} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} u(j\omega)^T u(j\omega) d\omega \right)^{1/2}$$

$$= \max_\omega \bar{\sigma}(G(j\omega)) = \|G(s)\|_{\infty} \quad \text{(alternative notation)}$$

$H_\infty$ is the set of stable, $H_\infty$-norm bounded transfer functions.

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The $H_2$ norm is another measure of the “size” or “gain” of a system.

$$\|G(s)\|_{H_2} := \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace} (G(j\omega)^* G(j\omega)) \ d\omega \right)^{1/2}$$

The integrand is the Frobenius norm squared of the frequency response:

$$\text{trace} (G(j\omega)^* G(j\omega)) = \sum_{i,j} |G_{ij}(j\omega)|^2 = \|G(j\omega)\|^2_F.$$

Via Parseval’s theorem:

$$\|G(s)\|_{H_2} = \|g(t)\|_{H_2} = \left( \int_0^{\infty} \text{trace} (g(\tau)^T g(\tau)) \ d\tau \right)^{1/2}$$
$H_2$ norm

For state-space representations:

$$\|G(s)\|_{H_2}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace} \left( G(j\omega)^* G(j\omega) \right) d\omega$$

$$= \int_0^\infty \text{trace} \left( B^T e^{A^T \tau} C^T Ce^{A\tau} B \right) d\tau$$

$$= \text{trace}(B^T W_o B) \quad (W_o: \text{observability Grammian})$$

$$= \text{trace}(CW_c C^T) \quad (W_c: \text{controllability Grammian})$$

(writing $\|G(s)\|_{H_2}^2$ avoids square roots)

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Nominal performance norm tests

$$\|N(s)\|_{H_2} < 1 \text{ implies:}$$

- If $w(t) = \delta(t)$, then $\|e(t)\|_2 < 1$.
- If $\|w(t)\|_2 < 1$, then $\max_t |e(t)| < 1$.
- If $w(t)$ is unit variance white noise, the $\text{var}(e(t)) < 1$.

$$\|N(s)\|_{H_\infty} < 1 \text{ implies:}$$

- If $w(t) = \sin(\omega t)$ then, $\max_t |e(t)| < 1$.
- If $\|w(t)\|_2 < 1$ then, $\|e(t)\|_2 < 1$. 
System norm comparison

\[ H_2 \] norm

- Useful nominal performance measure.
- Linear quadratic (LQ) design methods use this norm.
- Minimizes “average” errors.

\[ H_\infty \] norm

- Useful nominal performance measure.
- Minimizes “worst-case” errors.
- Induced norm: small-gain applies.
- Very useful for robustness analysis.

Notes and references

Skogestad & Postlethwaite (2nd Ed.)

- Internal stability: section 4.7
- Stabilizing controllers: section 4.8
- Stability analysis: section 4.9
- System norms: section 4.10

IMC design


MIMO Nyquist criterion