MIMO analysis: loop-at-a-time

Plant:

\[ P(s) = \frac{1}{s^2 + \alpha^2} \begin{bmatrix} s - \alpha^2 & \alpha(s + 1) \\ -\alpha(s + 1) & s - \alpha^2 \end{bmatrix} \]. (take \( \alpha = 10 \) in the following numerical analysis)

Controller:

\[ K_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad K_2 = \frac{1}{1 + \alpha^2} \begin{bmatrix} 1 & -\alpha \\ \alpha & 1 \end{bmatrix} \]
MIMO analysis: loop-at-a-time

Closed-loop transfer function:

\[
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix}
= P(s) \left( I + P(s)K_1(s) \right)^{-1} K_2(s) \begin{bmatrix}
r_1 \\
r_2
\end{bmatrix}
\]

\[
= \frac{1}{s + 1} \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
r_1 \\
r_2
\end{bmatrix}.
\]
MIMO analysis: loop-at-a-time

Loop transfer function: \[ e_1 = \frac{1}{s} u_1. \]
MIMO analysis

More general input perturbation analysis:

Perturbations at the plant input (actuator uncertainty)

\[ u_1 = (1 + \delta_1) e_1. \]
\[ u_2 = (1 + \delta_2) e_2 \]

\[ \Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}, \]
MIMO analysis

\[
\Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix},
\]

Take \( \|\Delta\| \leq 0.05 \) \hspace{1cm} (5\% \text{ uncertainty in each actuator})

Step response for \( y_1 \) channel

\[
\Delta = \begin{bmatrix} 0.05 & 0 \\ 0 & -0.05 \end{bmatrix},
\]

\( y_1 \) \hspace{1cm} \( y_2 \)

\( \delta_1 = 0.05, \delta_2 = -0.05 \)
MIMO analysis

The closed-loop system becomes unstable with \( \Delta = \begin{bmatrix} 0.11 & 0 \\ 0 & -0.11 \end{bmatrix} \),

Closed-loop characteristic polynomial:

\[
s^2 + (2 + \delta_1 + \delta_2) s + [1 + \delta_1 + \delta_2 + (\alpha^2 + 1)\delta_1\delta_2] = 0.
\]

If \( \delta_2 = 0 \) the smallest destabilizing perturbation is \( \delta_1 = -1 \)

Choose instead: \( \delta_1 = \frac{1}{\sqrt{\alpha^2 + 1}} \approx 0.1 \) and \( \delta_2 = -\delta_1 \)
MIMO analysis

Stability regions:

- **UNSTABLE**
- **STABLE**

\( \delta_1 \quad \delta_2 \)

\( -2 \quad -1 \quad 0 \quad 1 \quad 2 \)

\( -2 \quad -1.5 \quad -1 \quad -0.5 \quad 0 \)

\( -1.5 \quad -1 \quad -0.5 \quad 0 \quad 0.5 \quad 1 \quad 1.5 \quad 2 \)
Singular value analysis

\[ \sigma(L) \leq 1 \text{ for all } \omega. \]  
Performance poor in low gain direction.

\[ \kappa(L) \gg 1. \]  
Very sensitive to errors in direction.
Motivating robustness example no. 2: Distillation Process [3.7.2]

Idealized dynamic model of a distillation column,

\[ G(s) = \frac{1}{75s + 1} \begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix} \] \quad (6.49)

(time is in minutes).

Figure 62: Response with decoupling controller to filtered reference input \( r_1 = 1/(5s + 1) \). The perturbed plant has 20% gain uncertainty as given by (6.52).

Inverse-based controller or equivalently steady-state decoupler with a PI controller \((k_1 = 0.7)\)

\[ K_{\text{inv}}(s) = \frac{k_1}{s} G^{-1}(s) = \frac{k_1 (1 + 75s)}{s} \begin{bmatrix} 0.3994 & -0.3149 \\ 0.3943 & -0.3200 \end{bmatrix} \]
Nominal performance (NP).

\[ K_{\text{inv}}(s) = \frac{k_1}{s} G^{-1}(s) = \frac{k_1(1 + 75s)}{s} \begin{bmatrix} 0.3994 & -0.3149 \\ 0.3943 & -0.3200 \end{bmatrix} \]

\[ GK_{\text{inv}} = K_{\text{inv}}G = \frac{0.7}{s} I \]

first order response with time constant 1.43 (Fig. 62).

Nominal performance (NP) achieved with decoupling controller.
Robust stability (RS).

\[ S = S_I = \frac{s}{s + 0.7}I; \quad T = T_I = \frac{1}{1.43s + 1} \]  \hspace{1cm} (6.51)

In each channel: GM=∞, PM=90°.

Input gain uncertainty (6.48) with \( \epsilon_1 = 0.2 \) and \( \epsilon_2 = -0.2 \):

\[ u'_1 = 1.2u_1, \quad u'_2 = 0.8u_2 \]  \hspace{1cm} (6.52)

\[ L'_I(s) = K_{\text{inv}}G' = K_{\text{inv}}G \begin{bmatrix} 1 + \epsilon_1 & 0 \\ 0 & 1 + \epsilon_2 \end{bmatrix} = \begin{bmatrix} 0.7 & 0 \\ \frac{s}{1 + \epsilon_1} & 1 + \epsilon_2 \end{bmatrix} \]  \hspace{1cm} (6.53)

Perturbed closed-loop poles are

\[ s_1 = -0.7(1 + \epsilon_1), \quad s_2 = -0.7(1 + \epsilon_2) \]  \hspace{1cm} (6.54)

Closed-loop stability as long as the input gains \( 1 + \epsilon_1 \) and \( 1 + \epsilon_2 \) remain positive

\[ \Rightarrow \text{Robust stability (RS) achieved with respect to input gain errors for the decoupling controller.} \]
Robust performance (RP).

Performance with model error poor (Fig. 62)

- SISO: NP+RS ⇒ RP (within a factor of 2)
- MIMO: NP+RS \(\not\Rightarrow\) RP
  (arbitrarily large violation)

RP is not achieved by the decoupling controller.

6.3.3 Robustness conclusions [3.7.3]

Multivariable plants can display a sensitivity to uncertainty (in this case input uncertainty) which is fundamentally different from what is possible in SISO systems.
General control problem formulation (3.8 in S&P)

Figure 63: General control configuration for the case with no model uncertainty

The overall control objective is to minimize some norm of the transfer function from $w$ to $z$, for example, the $\mathcal{H}_\infty$ norm. The controller design problem is then:

Find a controller $K$ which based on the information in $v$, generates a control signal $u$ which counteracts the influence of $w$ on $z$, thereby minimizing the closed-loop norm from $w$ to $z$. 
Calculating the generalized plant

The routines in MATLAB for synthesizing $\mathcal{H}_\infty$ and $\mathcal{H}_2$ optimal controllers assume that the problem is in the general form of Figure 63

**Example: One degree-of-freedom feedback control configuration.**

![Figure 64: One degree-of-freedom control configuration](image)

Figure 64: One degree-of-freedom control configuration

The overall control objective is to minimize some norm of the transfer function from $w$ to $z$, for example, the $\mathcal{H}_\infty$ norm. The controller design problem is then: Find a controller $K$ which based on the information in $v$, generate a control signal $u$ which counteracts the influence of $w$ on $z$, thereby minimizing the closed-loop norm from $w$ to $z$. For more details, refer to Figure 64.
Calculating the generalized plant

Equivalent representation of Figure 64 where the error signal to be minimized is $z = y - r$ and the input to the controller is $v = r - y_m$

Figure 65: General control configuration equivalent to Figure 64
Calculating the generalized plant

\[
\begin{align*}
    w &= \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} d \\ r \\ n \end{bmatrix};
    z &= e = y - r; 
    v = r - y_m = r - y - n \\
    \text{(6.55)}
\end{align*}
\]

\[
\begin{align*}
    z &= y - r = Gu + d - r = Iw_1 - Iw_2 + 0w_3 + Gu \\
    v &= r - y_m = r - Gu - d - n = \\
    &= -Iw_1 + Iw_2 - Iw_3 - Gu
\end{align*}
\]

and \( P \) which represents the transfer function matrix from \( \begin{bmatrix} w & u \end{bmatrix}^T \) to \( \begin{bmatrix} z & v \end{bmatrix}^T \) is

\[
P = \begin{bmatrix}
    I & -I & 0 & G \\
    -I & I & -I & -G
\end{bmatrix}
\]

\[
\text{(6.56)}
\]

Note that \( P \) does not contain the controller.
Alternatively, \( P \) can be obtained from Figure 65.
Calculating the generalized plant

**Remark.** In MATLAB we may obtain $P$ via `simulink`, or we may use the `sysic` program in the Robust Control toolbox. The code in Table 2 generates the generalized plant $P$ in (6.56) for Figure 64.

### Table 2: Matlab program to generate $P$

<table>
<thead>
<tr>
<th>Code</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>% Uses the Robust Control toolbox</code></td>
<td></td>
</tr>
<tr>
<td><code>systemnames = 'G';</code></td>
<td>% $G$ is the SISO plant.</td>
</tr>
<tr>
<td><code>inputvar = '[d(1);r(1);n(1);u(1)];'</code></td>
<td>% Consists of vectors $w$ and $u$.</td>
</tr>
<tr>
<td><code>inputtoG = '[u]';</code></td>
<td></td>
</tr>
<tr>
<td><code>outputvar = '[G+d-r; r-G-d-n]';</code></td>
<td>% Consists of vectors $z$ and $v$.</td>
</tr>
<tr>
<td><code>sysoutname = 'P';</code></td>
<td></td>
</tr>
<tr>
<td><code>sysic;</code></td>
<td></td>
</tr>
</tbody>
</table>
Calculating the generalized plant
Calculating the generalized plant

\[ z = y - r \]

\[ v = r - y \]

\[ P \]

Figure 65: General control configuration equivalent to Figure 64

\[
\begin{align*}
[A, B, C, D] &= \text{linmod}('P\_diagram'); \\
P &= \text{ss}(A, B, C, D); \\
P &= \text{minreal}(P);
\end{align*}
\]
Including weights

To get a meaningful controller synthesis problem, for example, in terms of the \( \mathcal{H}_\infty \) or \( \mathcal{H}_2 \) norms, we generally have to include weights \( W_z \) and \( W_w \) in the generalized plant \( P \), see Figure 66.

That is, we consider the weighted or normalized exogenous inputs \( w \), and the weighted or normalized controlled outputs \( z = W_z \tilde{z} \). The weighting matrices are usually frequency dependent and typically selected such that weighted signals \( w \) and \( z \) are of magnitude 1, that is, the norm from \( w \) to \( z \) should be less than 1.

Figure 66: General control configuration for the case with no model uncertainty
**Example: Stacked S/T/KS problem.**

Consider an $\mathcal{H}_\infty$ problem where we want to bound $\bar{\sigma}(S)$ (for performance), $\bar{\sigma}(T)$ (for robustness and to avoid sensitivity to noise) and $\bar{\sigma}(KS)$ (to penalize large inputs). These requirements may be combined into a stacked $\mathcal{H}_\infty$ problem

$$\min_{K} \|N(K)\|_{\infty}, \quad N = \begin{bmatrix} W_u KS \\ W_T T \\ W_P S \end{bmatrix}$$

where $K$ is a stabilizing controller. In other words, we have $z = Nw$ and the objective is to minimize the $\mathcal{H}_\infty$ norm from $w$ to $z$. 

(6.57)
Figure 67: Block diagram corresponding to generalized plant in (6.57)

\[ z_1 = W_u u \]
\[ z_2 = W_T G u \]
\[ z_3 = W_P w + W_P G u \]
\[ v = -w - G u \]

so the generalized plant \( P \) from \([ w \quad u]^T\) to \([ z \quad v]^T\) is

\[
P = \begin{bmatrix}
0 & W_u I \\
0 & W_T G \\
W_P I & W_P G \\
-I & -G
\end{bmatrix}
\] (6.58)
6.4.3 Partitioning the generalized plant $P$

We often partition $P$ as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$  \hspace{1cm} (6.59)$$

so that

$$z = P_{11}w + P_{12}u$$  \hspace{1cm} (6.60)$$
$$v = P_{21}w + P_{22}u$$  \hspace{1cm} (6.61)$$

In Example “Stacked S/T/KS problem” we get from (6.58)

$$P_{11} = \begin{bmatrix} 0 \\ W_P I \end{bmatrix}, \quad P_{12} = \begin{bmatrix} W_u I \\ W_T G \end{bmatrix}$$  \hspace{1cm} (6.62)$$

$$P_{21} = -I, \quad P_{22} = -G$$  \hspace{1cm} (6.63)$$

Note that $P_{22}$ has dimensions compatible with the controller $K$ in Figure 6-35.
6.4.3 Partitioning the generalized plant $P$

[3.8.3]

We often partition $P$ as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

so that

$$z = P_{11}w + P_{12}u \quad (6.60)$$

$$v = P_{21}w + P_{22}u \quad (6.61)$$

In Example “Stacked $S/T/KS$ problem” we get from (6.58)

$$P_{11} = \begin{bmatrix} 0 \\ 0 \\ W_PI \end{bmatrix}, \quad P_{12} = \begin{bmatrix} W_uI \\ W_TG \\ W_PG \end{bmatrix} \quad (6.62)$$

$$P_{21} = -I, \quad P_{22} = -G \quad (6.63)$$

Note that $P_{22}$ has dimensions compatible with the controller $K$ in Figure 66
Calculating the closed-loop:

\[ w \rightarrow N \rightarrow z \]

Figure 68: General block diagram for analysis with no uncertainty

For analysis of closed-loop performance we may absorb \( K \) into the interconnection structure and obtain the system \( N \) as shown in Figure 68 where

\[ z = Nw \quad (6.64) \]

where \( N \) is a function of \( K \). To find \( N \), first partition the generalized plant \( P \) as given in (6.59)-(6.61), combine this with the controller equation

\[ u = Kv \quad (6.65) \]

and eliminate \( u \) and \( v \) from equations (6.60), (6.61) and (6.65) to yield \( z = Nw \) where \( N \) is given by

\[ N = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \triangleq F_i(P,K) \quad (6.66) \]
Calculating the closed-loop:

\[ N = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \triangleq F_l(P, K) \]

Here \( F_l(P, K) \) denotes a lower linear fractional transformation (LFT) of \( P \) with \( K \) as the parameter. In words, \( N \) is obtained from Figure 63 by using \( K \) to close a lower feedback loop around \( P \). Since positive feedback is used in the general configuration in Figure 63 the term \((I - P_{22}K)^{-1}\) has a negative sign.
Robustness: a more general control structure

The general control configuration in Figure 63 may be extended to include model uncertainty. Here the matrix $\Delta$ is a *block-diagonal* matrix that includes all possible perturbations (representing uncertainty) to the system. It is normalized such that $\|\Delta\|_\infty \leq 1$.

![General control configuration for the case with model uncertainty](image)

**SISO example: input multiplicative**

![Plant with multiplicative uncertainty](image)

Figure 33: Plant with multiplicative uncertainty
Robustness: a more general control structure

Figure 71: General block diagram for analysis with uncertainty included

Figure 72: Rearranging a system with multiple perturbations into the $N\Delta$-structure
Robustness: a more general control structure

The block diagram in Figure 70 in terms of $P$ (for synthesis) may be transformed into the block diagram in Figure 71 in terms of $N$ (for analysis) by using $K$ to close a lower loop around $P$. The same lower LFT as found in (6.66) applies, and

$$N = F_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \quad (6.71)$$

To evaluate the perturbed (uncertain) transfer function from external inputs $w$ to external outputs $z$, we use $\Delta$ to close the upper loop around $N$ (see Figure 71), resulting in an upper LFT:

$$z = F_u(N, \Delta)w; \quad (6.72)$$

$$F_u(N, \Delta) \triangleq N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12} \quad (6.73)$$