Digital implementation of control systems

Why digital?
- Easily reprogrammed or modified.
- Complex algorithms (or optimisations) can be implemented.
- Integration with remote systems (via internet).

Why analogue?
- Simple and cheap in mass production.
- Highly reliable.
- Very high frequency operation.
- On-chip integrated systems.
Sampled-data control systems

Components:
- Plant: $G(s)$, continuous-time;
- Controller: $K_d(z)$, discrete-time;
- Sampler (A/D converter): $y(k) = y(t)|_{t=kT}$ for $k = 0, 1, 2, \ldots$
- Zero-order hold (D/A converter): $u(t) = u(kT)$, $kT \leq t < kT + T$.

Components: sampler

$y(k) = y(t)|_{t=kT}$, $k = 0, 1, 2, \ldots$

$T$ is the sampling period.
Components: zero-order hold

\[ u(t) = u(k), \quad \text{for } kT \leq t < kT + T. \]

Discrete sequence:

Continuous signal:

Quantization

Potential error: \( \pm 1/2 \) LSB in the best case.

Example: 12 bit A/D and D/A on a ±10 volt scale
1 LSB = 0.00488 volts.
Sampled-data reconstruction

\[ \tilde{x}(t) \xrightarrow{ZOH} x(k) \xrightarrow{T} x(t) \]

Input signal: \( x(t) \)

Output signal: \( \tilde{x}(t) \)

Sampling

\[ y(k) \xrightarrow{T} y(t) \]

Example: single pole signal

Consider \( y(t) = \begin{cases} e^{-at}, & t \geq 0 \\ 0, & t < 0 \end{cases} \) with \( a > 0 \).

Laplace transform: \( y(s) = \frac{1}{s + a} \).

Sampled signal: \( y(k) = \left. y(t) \right|_{t=kT} = e^{-akT} = \left( e^{-aT} \right)^k \).

Z-transform: \( y(z) = \frac{z}{z - e^{-aT}} \).

The \( s \)-plane pole is at \( s_1 = -a \), and the corresponding \( z \)-plane pole is at \( z_1 = e^{-aT} \).
Sampling

General case:

Sampling maps the $s$-domain poles to the $z$-domain via: $z_i = e^{s_i T}$.
Stability preserving: \{real$(s_i) < 0$\} maps to \{|$z_i| < 1$\}.

Pole locations under sampling:

![Diagram showing pole locations under sampling]

Sampling (in detail)

$Z$-plane

$\omega_n = 0.6\pi/T \quad N = 4 \quad \omega_n = 0.5\pi/T$
$\omega_n = 0.7\pi/T \quad N = 3 \quad \omega_n = 0.4\pi/T$
$\omega_n = 0.8\pi/T \quad N = 2 \quad \omega_n = 0.3\pi/T$
$\omega_n = 0.9\pi/T$

Lines of constant frequency
Lines of constant damping
Samples per oscillation

Changing the sampling frequency.

Decreasing $T$: decrease decay rate ($r \to 1$)
decrease oscillation frequency ($\theta \to 0$)
poles track constant damping curves towards 1
Sampled signals: aliasing

Example:

55 Hz signal sampled at 275 Hz

Consider \( y(t) = \cos \omega_1 t \)

Laplace: \( y(s) = \frac{s}{s^2 + \omega_1^2} \).

Continuous poles: \( s_{1,2} = \pm j\omega_1 \)

Sampled poles: \( z_{1,2} = e^{\pm j\omega_1 T} \)
Sampled signals: aliasing

Example:

55 Hz signal sampled at: \( \frac{1}{T_1} = 275 \) Hz
\( \frac{1}{T_2} = 130 \) Hz
\( \frac{1}{T_3} = 65 \) Hz

Real
Imaginary
Z-plane

-1 1

ω1T1

Amplitude

-1.5
-1
-0.5
0
0.5
1.0
1.5

Time (seconds)

0.1 0.2

2015-5-14 12.13

Sampled signals: aliasing

Example:

55 Hz signal sampled at: \( \frac{1}{T_1} = 275 \) Hz
\( \frac{1}{T_2} = 130 \) Hz

Real
Imaginary
Z-plane

-1 1

ω1T2

Amplitude

-1.5
-1
-0.5
0
0.5
1.0
1.5

Time (seconds)

0.1 0.2

2015-5-14 12.13
Sampled signals: aliasing

Example:

55 Hz signal sampled at:
\[ \frac{1}{T_1} = 275 \text{ Hz} \]
\[ \frac{1}{T_2} = 130 \text{ Hz} \]
\[ \frac{1}{T_3} = 65 \text{ Hz} \]
Sampled signals: aliasing

The unit disk can only represent signals of frequency up to 1/2 the sampling frequency. (Nyquist frequency).

Maps the horizontal strip from $-j\pi/T$ to $j\pi/T$ onto the $z$-plane.

And $\text{real}(s) < 0$ in this strip maps to the inside of the unit disk.

Sampling also maps the next strip (from $j\pi/T$ to $j3\pi/T$) onto the whole $z$-plane and adds it into the result.

Also true for all (infinite) $2\pi/T$ wide strips above and below the lowest frequency strip.
Aliased high frequency disturbances are indistinguishable from low frequency disturbances.

The controller responds at the wrong frequency.

Consequences of aliasing:

- Ambiguity. Our computer/controller cannot distinguish between frequencies inside the $-\pi/T$ to $\pi/T$ range and those outside of it.
  - Controller will respond incorrectly to an aliased signal.
  - An aliased signal cannot be reconstructed (signal processing).

Amelioration of the problem:

- Anti-aliasing filter. Low pass, rejecting $|\omega| > \pi/T$.
  - High frequency signals no longer enter loop erroneously.
  - High frequency disturbances/errors are "invisible."
Anti-aliasing filters

The anti-aliasing filter, $F_a(s)$, will add phase to the loop (Potentially destabilizing!)

ZOH response

Pulse input: $u(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$, gives the output,

Equivalently, the pulse response is:

\[ u(t) = \text{step}(t) - \text{step}(t - T), \quad (\text{step}(t) = \text{unit step function}) \]
The discrete-time transfer function is the $z$-transform of the sampled pulse response.

For a pulse, $u(k)$, the plant input is,

$$u(t) = \text{step}(t) - \text{step}(t - T).$$

The ZOH output (in the Laplace domain) is

$$u(s) = \left(1 - e^{-Ts}\right) \frac{1}{s}.$$
**ZOH properties**

**Magnitude**

![Magnitude plot with ZOH(jω) and Ideal curves]

**Phase (degrees)**

![Phase plot with ZOH(jω) and Ideal curves]

**ZOH properties**

\[ x(t) \xrightarrow{\text{ZOH}} x(k) \xrightarrow{T} \tilde{x}(t) \]

**ZOH output**

![ZOH output and 1st harmonic with Input sinusoid and Samples]

**Time [seconds]**

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Zero-order hold equivalence

\[
\begin{align*}
\text{Input: } &\quad u(k) = \begin{cases} 
1 & k = 0, \\
0 & k \neq 0,
\end{cases} \quad u(t) = \text{step}(t) - \text{step}(t - T). \\
\text{Output: } &\quad y(s) = \left(1 - e^{-Ts}\right) \frac{G(s)}{s}. 
\end{align*}
\]

We now sample this, and take the \(Z\)-transform,

\[
G_{\text{ZOH}}(z) = \mathcal{Z} \left\{ \left(1 - e^{-Ts}\right) \frac{G(s)}{s} \right\} = (1 - z^{-1}) \mathcal{Z} \left\{ \frac{G(s)}{s} \right\}.
\]

Easily calculated (c2d or zohequiv in MATLAB).

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Sampling

Sample rate effects:

\[
z_i = e^{s_i T}, \text{ changing } T \text{ changes the pole positions.}
\]

Continuous closed-loop step response:

\[
G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\]

\[
\zeta = 0.6, \quad \omega_n = 5 \text{ rad./sec.,} \\
\omega_n = -3 \pm 4i.
\]

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Sampling

Sample rate effects:
Continuous pole positions: $s_{1,2} = -3 \pm 4i$.

Sample period $T_1$: $z_{1,2} = 0.255 \pm 0.398i$
Sample period $T_2$: $z_{1,2} = 0.650 \pm 0.306i$

Continuous time: root-locus analysis of closed-loop stability

Plant:
$$G(s) = \frac{a}{s+a}, \quad a > 0.$$  

Controller (proportional):
$$K(s) = K_p.$$  

$G(s)K(s)$ has one pole and no zeros.

Theoretically stable for $-1 < K_p \leq \infty$. 

Discrete time: root-locus analysis of closed-loop stability

![Diagram of discrete time system with root-locus analysis]

The additional phase from the ZOH is also potentially destabilizing.

\[ G_{ZOH}(z) = \frac{1 - e^{-aT}}{z - e^{-aT}} \]

\[ G_{ZOH}(z)K \] has one pole and no zeros.

Unstable for \( K > \frac{1 + e^{-aT}}{1 - e^{-aT}} \)