Digital control system design

Sampled-data closed-loop

\[ y(k) \rightarrow T \rightarrow y(t) \rightarrow G(s) \rightarrow ZOH \rightarrow u(t) \rightarrow K_d(z) \rightarrow + \rightarrow r(k) \]

\[ G_ZOH(z) \] equivalence

\[ y(k) \rightarrow T \rightarrow y(t) \rightarrow G(s) \rightarrow ZOH \rightarrow K_d(z) \rightarrow + \rightarrow r(k) \]
Zero-order hold equivalence — transfer function

\[ y(k) \xrightarrow{T} y(t) \xrightarrow{G(s)} u(t) \xrightarrow{\text{ZOH}} u(k) \]

\[ G_{\text{ZOH}}(z) \]

Input: \( u(k) = \begin{cases} 1 & k = 0, \\ 0 & k \neq 0 \end{cases}, \quad u(t) = \text{step}(t) - \text{step}(t - T). \)

Output: \( y(s) = \left(1 - e^{-Ts}\right) \frac{G(s)}{s}. \)

We now sample this, and take the \( Z \)-transform,

\[ G_{\text{ZOH}}(z) = Z \left\{ \left(1 - e^{-Ts}\right) \frac{G(s)}{s} \right\} = (1 - z^{-1}) Z \left\{ \frac{G(s)}{s} \right\}. \]

Zero-order hold equivalence — state space

Integrating \( \Phi(t) \) over a single sample period (\( kT \) to \( kT + T \)):

\[ x(kT + T) = e^{AT} x(kT) + \int_{kT}^{kT+T} e^{A(kT+T-\tau)} B u(\tau) d\tau, \]

\[ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \xrightarrow{\text{ZOH-equivalence}} \begin{bmatrix} e^{AT} & \int_0^T e^{A\eta} B d\eta \\ C & D \end{bmatrix} \]
Zero-order hold equivalence — frequency domain

Example: \[ G(s) = \frac{(2 - s)}{(2s + 1)(s + 2)} , \quad T = 0.6 \]

Magnitude
\[ G(j\omega) \]
\[ G(j\omega)e^{-j\omega T/2} \]
\[ G_{ZOH}(e^{j\omega}) \]

Phase (deg.)
\[ G(j\omega) \]
\[ G(j\omega)e^{-j\omega T/2} \]
\[ G_{ZOH}(e^{j\omega}) \]

Digital control system design

Sampled-data closed-loop

\[ G(s) \]
\[ T \]
\[ ZOH \]
\[ K_d(z) \]
\[ r(k) \]
\[ u(t) \]
\[ u(k) \]

\[ K(s) \] approximation

\[ \approx K(s) \]
Design approaches

Continuous-time design of $G(s)$ and sample/hold

Discrete-time design

Approximation of $K(s)$ with $K_d(z)$

$ZOH$-equivalence of $G(s)$ and sample/hold

Sampled-data design
Design by approximation

1. Design a continuous-time controller, $K(s)$
   - Verify stability, performance and bandwidth
   - Verify margins and robustness

2. Select a sample-rate, $T$

3. Find $K_d(z)$ approximating $K(s)$

4. Calculate the ZOH-equivalent $G_{ZOH}(z)$

5. Check the stability of the $G_{ZOH}(z), K_d(z)$ loop

6. Simulate $K_d(z)$ with $G(s)$ (including sample/hold).
   - Verify simulated performance
   - Examine intersample behaviour

Controller approximation

Approach: approximating the integrators

If $F(z) \approx \frac{1}{s}$, then, $s \approx F^{-1}(z)$, \( \implies K_d(z) = K(s) \bigg|_{s=F^{-1}(z)} \)
**Integration**

\[ y(t) = y(0) + \int_0^t x(\tau) \, d\tau, \]

The signal, \( y(t) \), over a single \( T \) second sample period is,

\[ y(kT + T) = y(kT) + \int_{kT}^{kT+T} x(\tau) \, d\tau. \]

**Trapezoidal approximation**

\[ \hat{y}(kT + T) = \hat{y}(kT) + Tx(kT) + (x(kT + T) - x(kT))T/2. \]

Taking \( z \)-transforms,

\[ z\hat{y}(z) = \hat{y}(z) + T x(z) + \frac{T}{2}(z - 1)x(z), \]

Approximation:

\[ \frac{\hat{y}(z)}{x(z)} = F(z) = \frac{T \, z + 1}{2 \, z - 1}. \]

So the substitution is,

\[ s \leftarrow \frac{2 \, z - 1}{T \, z + 1}. \]

This is known as a bilinear (or Tustin) transform.
Frequency mapping

Pole locations under bilinear transform:

\[
\{ s | \text{real}(s) < 0 \} \quad \text{bilinear} \quad \rightarrow \quad \{ z | |z| < 1 \}
\]

\[ K(s) \text{ stable} \iff K_d(z) \text{ stable.} \]

Bilinear frequency distortion

\( \Omega : \) discrete-frequencies: \( e^{j\Omega T} \), \( \Omega \in (-\pi, \pi] \).

Frequency mapping:

Continuous frequencies, \( \omega \) to discrete frequencies, \( \Omega \).

Substitute \( s = j\omega \) and \( z = e^{j\Omega T} \) into \( s = \frac{2}{T} \frac{z - 1}{z + 1} \):

\[
j\omega = \frac{2}{T} \frac{1 - e^{-j\Omega T}}{1 + e^{-j\Omega T}} = \frac{2}{T} \frac{j \sin(\Omega T/2)}{\cos(\Omega T/2)} = \frac{2}{T} j \tan(\Omega T/2).
\]

Frequency distortion:

\[ \Omega = \frac{2}{T} \tan^{-1}(\omega T/2) \]
Bilinear frequency distortion

\[ \Omega = \frac{2}{T} \tan^{-1} \left( \frac{\omega T}{2} \right) \]

Discrete frequency (\(\Omega\)): \([\text{rad/sec}]\)
Continuous frequency (\(\omega\)): \([\text{rad/sec}]\)

The \(\Omega = \omega T\) line is the sampling mapping.

Prewarping

\[ s = \frac{\alpha(z - 1)}{(z + 1)} \], \(\alpha \in (0, \pi/T)\), maps \(|\text{real}\{s\}| < 0\) to \(|z| < 1\).

Modifying the frequency distortion

Select a frequency \(\omega_{pw}\).

Solve for \(\alpha\) such that \(K(j\omega_{pw}) = K_d(e^{j\omega_{pw}T})\).

The “prewarped” transform makes \(K(j\omega) = K_d(e^{j\omega T})\) at \(\omega = 0\) and \(\omega = \omega_{pw}\).

\[ j\omega_0 = \frac{\alpha(e^{j\omega_{pw}T} - 1)}{(e^{j\omega_{pw}T} + 1)} = j\alpha \tan(\omega_{pw}T/2), \]

which implies that: \(\alpha = \frac{\omega_{pw}}{\tan(\omega_{pw}T/2)}\).
**Prewarping**

**Frequency distortion (bilinear):** \( \Omega = \frac{2}{T} \tan^{-1}(\omega T/2) \).

**Frequency distortion (with prewarping):** \( \Omega = \frac{2}{T} \tan^{-1}(\omega/\alpha) \)

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**Controller approximations**

**Bilinear approximation:** \( K_{bl}(e^{j\omega}) \), **Prewarped Tustin approximation:** \( K_{pw}(e^{j\omega}) \)
Choosing a prewarping frequency

The prewarping frequency must be in the range: \(0 < \omega_{pw} < \pi/T\).

- \(\alpha = 2/T\) (standard bilinear) corresponds to \(\omega_{pw} = 0\).

Possible options for \(\omega_{pw}\) depend on the problem:

- The cross-over frequency (helps preserve the phase margin);
- The frequency of a critical notch;
- The frequency of a critical oscillatory mode.

The best choice depends on the most important features in your control design.

**Remember:** \(K(s)\) stable implies \(K_d(z)\) stable.

But you **must** check that \((1 + G_{ZOH}(z)K_d(z))^{-1}\) is stable!

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**Example**

**Plant model:**

\[
G(s) = \frac{5(1 - s/z_{rhp})}{(1 + \tau s)} \left( \frac{(s^2 + 2\zeta\eta\omega_m s + \eta^2\omega_m^2)}{(s^2 + 2\zeta\omega_m s + \omega_m^2)} \right) \frac{1}{\eta^2}
\]

where \(\tau = 0.5\), \(z_{rhp} = 70\), \(\omega_m = 20\), \(\zeta = 0.05\), and \(\eta = 1.2\).

**IMC design**

\[
T_{ideal}(s) = 3rd \text{ order Butterworth filter with bandwidth: } 25 \text{ [rad./sec.]}
\]

\[
Q(s) = T_{ideal}(s)G_{mp}^{-1}(s)
\]

\[
K(s) = (I - Q(s)G(s))^{-1}Q(s).
\]
Loopshapes

Sensitivity and complementary sensitivity
Step response

Output

\[
\begin{align*}
y(t) &= -0.2, 0, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4 \\
time (sec) &= 0.5, 1.0, 1.5, 2.0
\end{align*}
\]

Step response

Actuation

\[
\begin{align*}
u(t) &= -0.2, 0, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4 \\
time (sec) &= 0.5, 1.0, 1.5, 2.0
\end{align*}
\]
Bilinear/Trapezoidal/Tustin transform

Bilinear transform

Nyquist frequency: 100 radians/second \( \implies T = \frac{\pi}{100} \).

\[ K_{bl}(z) = K(s) \bigg|_{s=\frac{2}{T} \frac{z-1}{z+1}} \]

Discrete-time analysis

[Diagram of discrete-time analysis]

Loopshapes: bilinearly transformed controller

[Graph showing magnitude and phase plots]
Sensitivity and complementary sensitivity: bilinearly transformed controller

Step response: bilinearly transformed controller
Step response: bilinearly transformed controller

Actuation

Prewarped Tustin transform

Nyquist frequency: 100 radians/second \( \implies T = \frac{\pi}{100}. \)

Select the prewarping frequency at \( \omega_{pw} = \omega_m \) (20 radians/sec.).

\[
K_{pw}(z) = K(s) \bigg|_{s = \alpha} = \frac{z^{-1}}{z+1}
\]

where,

\[
\alpha = \frac{\omega_{pw}}{\tan(\omega_{pw}T/2)}.
\]
Loopshapes: prewarped Tustin controller

Sensitivity and complementary sensitivity: prewarped Tustin controller
Step response: prewarped Tustin controller

Output

Amplitude

\[ y(t) \]

\[ y(k) \]

-0.2
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4

Time (sec)

0.5 1.0 1.5 2.0

Actuation

Amplitude

\[ u(t) \]

\[ u(k) \]

-0.2
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4

Time (sec)

0.5 1.0 1.5 2.0
Sample rate selection

Sample rate selection is critical to digital control design.

Main considerations
▶ Sampling/ZOH will (approximately) introduce a delay of $T/2$ seconds.
▶ Anti-aliasing filters will need to be designed and these will also introduce phase lag.
▶ The system runs “open-loop” between samples.
▶ Very fast sampling can introduce additional noise.
▶ Very fast sampling makes all of the poles appear close to 1. The controller design can become numerically sensitive.

Designing for digital implementation

Sampled-data implementation

Continuous-time design
Sensitivity function

We want a similar discrete sensitivity function up to the frequency where $|S(j\omega)|$ returns to 1.

$$S(s) = \left( I + F_a(s)G(s)e^{-sT/2}K(s) \right)^{-1}$$

In this example, for $\omega > 20$ rad./sec.,

$$|1 - S(j\omega)| \ll 1 \implies \frac{\pi}{T} = 20 \text{ is about the minimum.}$$

$$S(s) = \left( I + F_a(s)G(s)e^{-sT/2}K(s) \right)^{-1}$$
Loop-shaping interpretation

For $\omega$ up to where $|F_a(j\omega)G(j\omega)K(j\omega)| < \epsilon$ and remains very small,
we want

$$F_a(j\omega)G(j\omega)K(j\omega)e^{-j\omega T/2} \approx \hat{G}_{ZOH}(e^{j\omega T})K_d(e^{j\omega T}).$$

($\hat{G}_{ZOH}(z)$ is the ZOH-equivalent of $F_a(s)G(s)$)

Fast sampling

Fast sampling period: $T_f$.

Control appropriate (slower) sampling period: $T_s$
(typically $T_s = MT_f$ for integer $M > 1$).