Digital control system design

Sampled-data closed-loop

\[ y(k) \quad \underbrace{\quad y(t) \quad}_{T} \quad G(s) \quad u(t) \quad ZOH \quad u(k) \quad K_d(z) \quad + \quad r(k) \]

\[ G_{ZOH}(z) \] equivalence

\[ y(k) \quad \underbrace{\quad y(t) \quad}_{T} \quad G(s) \quad ZOH \quad G_{ZOH}(z) \quad K_d(z) \quad + \quad r(k) \]
Zero-order hold equivalence — transfer function

\[
\begin{align*}
G(s) & = \left. \frac{y(t)}{u(t)} \right|_{t=kT} \\
\text{Input: } u(k) & = \begin{cases} 
1 & k = 0, \\
0 & k \neq 0,
\end{cases} \quad u(t) = \text{step}(t) - \text{step}(t - T). \\
\text{Output: } y(s) & = \left(1 - e^{-Ts} \right) \frac{G(s)}{s}.
\end{align*}
\]

We now sample this, and take the Z-transform,

\[
G_{ZOH}(z) = Z \left\{ \left(1 - e^{-Ts} \right) \frac{G(s)}{s} \right\} = (1 - z^{-1}) Z \left\{ \frac{G(s)}{s} \right\}.
\]

Zero-order hold equivalence — state space

Integrating Φ(t) over a single sample period (kT to kT + T):

\[
x(kT + T) = e^{AT} x(kT) + \int_{kT}^{kT+T} e^{A(kT+T-\tau)} Bu(\tau) d\tau,
\]

\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{ZOH-equivalence} \quad \Rightarrow \quad \begin{bmatrix} e^{AT} \\ \int_0^T e^{A\eta} B d\eta \end{bmatrix}
\]

\[
\begin{bmatrix} e^{AT} \\ \int_0^T e^{A\eta} B d\eta \end{bmatrix}
\]
Zero-order hold equivalence — frequency domain

Example: \( G(s) = \frac{(2 - s)}{(2s + 1)(s + 2)} \), \( T = 0.6 \)

\[ G_{ZOH}(e^{j\omega}) \]

\[ G(j\omega) = G(j\omega)e^{j\omega T/2} \]

Magnitude

Phase (deg.)

Digital control system design

Sampled-data closed-loop

\( y(k) \)

\( \frac{1}{T} \)

\( y(t) \)

\( G(s) \)

\( u(t) \)

\( u(k) \)

\( K_d(z) \)

\( r(k) \)

\( K(s) \) approximation

\( y(k) \)

\( \frac{1}{T} \)

\( G(s) \)

\( ZOH \)

\( K_d(z) \)

\( r(k) \)
Design approaches

$G(s)$ \xrightarrow{\text{Continuous-time design}} K(s)

$G_{ZOH}(z)$ \xrightarrow{\text{Discrete-time design}} K_d(z)

ZOH-equivalence of $G(s)$ and sample/hold

Approximation of $K(s)$ with $K_d(z)$

Sampled-data design

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**Design by approximation**

1. Design a continuous-time controller, $K(s)$
   - Verify stability, performance and bandwidth
   - Verify margins and robustness
2. Select a sample-rate, $T$
3. Find $K_d(z)$ approximating $K(s)$
4. Calculate the ZOH-equivalent $G_{ZOH}(z)$
5. Check the stability of the $G_{ZOH}(z), K_d(z)$ loop
6. Simulate $K_d(z)$ with $G(s)$ (including sample/hold).
   - Verify simulated performance
   - Examine intersample behaviour

**Controller approximation**

Approach: approximating the integrators

If $F(z) \approx 1/s$, then, $s \approx F^{-1}(z)$, \implies $K_d(z) = K(s) \mid_{s=F^{-1}(z)}$
Integration

\[ y(t) = \frac{1}{s} x(t) \]

\[ y(t) = y(0) + \int_0^t x(\tau) \, d\tau, \]

The signal, \( y(t) \), over a single \( T \) second sample period is,

\[ y(kT + T) = y(kT) + \int_{kT}^{kT+T} x(\tau) \, d\tau. \]

Trapezoidal approximation

\[ \hat{y}(kT + T) = \hat{y}(kT) + T x(kT) + (x(kT + T) - x(kT))T/2. \]

Taking \( z \)-transforms,

\[ z \hat{y}(z) = \hat{y}(z) + T x(z) + \frac{T}{2} (z - 1) x(z), \]

Approximation:

\[ \frac{\hat{y}(z)}{x(z)} = F(z) = \frac{T}{2} \frac{z + 1}{z - 1}. \]

The substitution is therefore,

\[ s \leftarrow \frac{2}{T} \frac{z - 1}{z + 1}. \]

This is known as a bilinear (or Tustin) transform.
Frequency mapping

Pole locations under bilinear transform:
\[ \{ s \mid \text{real}(s) < 0 \} \quad \xrightarrow{\text{bilinear}} \quad \{ z \mid |z| < 1 \} \]

\( K(s) \) stable \iff \( K_d(z) \) stable.

Bilinear frequency distortion

\( \Omega \): discrete-frequencies: \( e^{j\Omega T}, \Omega \in (-\pi, \pi] \).

Frequency mapping:

Continuous frequencies, \( \omega \) to discrete frequencies, \( \Omega \).

Substitute \( s = j\omega \) and \( z = e^{j\Omega T} \) into \( s = \frac{2}{T} \frac{z-1}{z+1} \):

\[
\frac{j\omega}{T} = \frac{2}{T} \frac{1 - e^{-j\Omega T}}{1 + e^{-j\Omega T}} = \frac{2}{T} \frac{j \sin(\Omega T/2)}{\cos(\Omega T/2)} = \frac{2}{T} j \tan(\Omega T/2).
\]

Frequency distortion:

\[ \Omega = \frac{2}{T} \tan^{-1}(\omega T/2) \]
Bilinear frequency distortion

\[ \Omega = \frac{2}{T} \tan^{-1}(\omega T/2) \]

Discrete frequency (\(\Omega\)): [rad/sec]

Continuous frequency (\(\omega\)): [rad/sec]

Tustin/bilinear transform

The \(\Omega = \omega T\) line is the sampling mapping.

Prewarping

\[ s = \frac{\alpha(z - 1)}{(z + 1)}, \ \alpha \in (0, \pi/T), \ \text{maps} \ \{\text{real}\{s\} < 0\} \ \text{to} \ \{|z| < 1\}. \]

Modifying the frequency distortion

Select a frequency \(\omega_0\).

Solve for \(\alpha\) such that \(K(j\omega_0) = K_d(e^{j\omega_0 T})\).

The "prewarped" transform makes \(K(j\omega) = K_d(e^{j\omega T})\) at \(\omega = 0\) and \(\omega = \omega_0\).

\[ j\omega_0 = \frac{\alpha(e^{j\omega_0 T} - 1)}{(e^{j\omega_0 T} + 1)} = j\alpha \tan(\omega_0 T/2), \]

which implies that: \(\alpha = \frac{\omega_0}{\tan(\omega_0 T/2)}\).
Prewarping

Frequency distortion (bilinear): \( \Omega = \frac{2}{T} \tan^{-1}(\omega T/2) \).

Frequency distortion (with prewarping): \( \Omega = \frac{2}{T} \tan^{-1}(\omega / \alpha) \).

Example

Compare bilinear and prewarped: \( \omega_0 = 50 \, \text{rad/sec.} \)
Choosing a prewarping frequency

The prewarping frequency must be in the range: $0 < \omega_0 < \pi/T$.

- $\alpha = 2/T$ (standard bilinear) corresponds to $\omega_0 = 0$.

Possible options for $\omega_0$ depend on the problem:

- The cross-over frequency (helps preserve the phase margin);
- The frequency of a critical notch;
- The frequency of a critical oscillatory mode.

The best choice depends on the most important features in your control design.

Remember: $K(s)$ stable implies $K_d(z)$ stable.

But you must check that $(1 + G_{ZOH}(z)K_d(z))^{-1}$ is stable!

Example

Plant model:

$$G(s) = \frac{5(1 - s/z_{rh})}{(1 + \tau s)} \left( \frac{2\zeta\omega_m s + \eta^2\omega_m^2}{s^2 + 2\zeta\omega_m s + \omega_m^2} \right) \frac{1}{\eta^2}$$

where $\tau = 0.5$, $z_{rh} = 70$, $\omega_m = 20$, $\zeta = 0.05$, and $\eta = 1.2$.

IMC design

$T_{\text{ideal}}(s) = 3$rd order Butterworth filter with bandwidth: 25 [rad./sec.]

$Q(s) = T_{\text{ideal}}(s)G_{mp}^{-1}(s)$

$K(s) = (I - Q(s)G(s))^{-1}Q(s)$. 

2014-5-25
Loopshapes

Magnitude

\[ L(j\omega) \]
\[ G(j\omega) \]
\[ K(j\omega) \]
\[ L(j\omega) \]
\[ G(j\omega) \]

Phase (deg.)

\[ \text{log} \omega \ (\text{rad/sec}) \]

Sensitivity and complementary sensitivity

Magnitude

\[ S(j\omega) \]
\[ T(j\omega) \]

\[ \text{log} \omega \ (\text{rad/sec}) \]

2014-5-25
Step response

Output

Amplitude

\[ y(t) \]

0.2 0.4 0.6 0.8 1.0 1.1 1.2 1.4

-0.2 0 0.2 0.4 0.6 0.8 1.0 1.2 1.4

0 0.5 1.0 1.5 2.0

time (sec)

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Step response

Actuation

Amplitude

\[ u(t) \]

0.2 0.4 0.6 0.8 1.0 1.2 1.4

-0.2 0 0.2 0.4 0.6 0.8 1.0 1.2 1.4

0 0.5 1.0 1.5 2.0

time (sec)

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Bilinear/Trapezoidal/Tustin transform

Bilinear transform

Nyquist frequency: 100 radians/second \( \implies T = \frac{\pi}{100}. \)

\[ K_d(z) = K(s) \bigg|_{s = \frac{2}{T} \frac{z-1}{z+1}} \]

Discrete-time analysis

\[ G_{ZOH}(z) \]

\[ y(k) \quad \underline{T} \quad G(s) \quad \underline{ZOH} \quad K_d(z) \quad \underline{+} \quad r(k) \]

Loopshapes: bilinear transformed controller

\[ \log \omega \text{ (rad/sec)} \]

\[ \text{Phase (deg.)} \]

\[ \text{Magnitude} \]

\[ K_d(e^{j\omega}) \quad G(j\omega) \quad G_{ZOH}(e^{j\omega}) \quad L_d(e^{j\omega}) \quad L(j\omega) \quad K_d(e^{j\omega}) \]
Sensitivity and complementary sensitivity: bilinear transformed controller

\[
S(j\omega) \quad T(j\omega) \quad S_d(e^{j\omega}) \quad T_d(e^{j\omega})
\]

Step response: bilinear transformed controller

Output
Step response: bilinear transformed controller

Actuation

Prewarped Tustin transform

Prewarped Tustin transform

Nyquist frequency: 100 radians/second $\implies T = \frac{\pi}{100}$.

Select the prewarping frequency at $\omega_0 = \omega_m$ (20 radians/sec.).

$$K_d(z) = K(s)\bigg|_{s = \alpha \frac{z-1}{z+1}}$$

where,

$$\alpha = \frac{\omega_0}{\tan(\omega_0 T/2)}.$$
Loopshapes: prewarped Tustin controller

Magnitude

Phase (deg.)

Sensitivity and complementary sensitivity: prewarped Tustin controller
Step response: prewarped Tustin controller

Output

Actuation
Sample rate selection

Sample rate selection is critical to digital control design.

Main considerations

- Sampling/ZOH will (approximately) introduce a delay of $T/2$ seconds.
- Anti-aliasing filters will need to be designed and these will also introduce phase lag.
- The system runs “open-loop” between samples.
- Very fast sampling can introduce additional noise.
- Very fast sampling makes all of the poles appear close to 1. The controller design can become numerically sensitive.

Designing for digital implementation

Sampled-data implementation

Continuous-time design
Sensitivity function

We want a similar discrete sensitivity function up to the frequency where $|S(j\omega)|$ returns to 1.

\[ S(s) = (I + F_a(s)G(s)e^{-sT/2}K(s))^{-1} \]

In this example, for $\omega > 20 \text{ rad./sec.}$,

\[ |1 - S(j\omega)| \ll 1 \implies \frac{\pi}{T} = 20 \text{ is about the minimum.} \]

\[ S(s) = (I + F_a(s)G(s)e^{-sT/2}K(s))^{-1} \]
Loop-shaping interpretation

For $\omega$ where $0.1 < |F_a(j\omega)G(j\omega)K(j\omega)| < 10$, we want

$$F_a(j\omega)G(j\omega)K(j\omega) \approx \hat{G}_{ZOH} \left( e^{j\omega T} \right) K_d \left( e^{j\omega T} \right).$$

($\hat{G}_{ZOH}(z)$ is the ZOH-equivalent of $F_a(s)G(s)$)

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Fast sampling

Fast sampling period: $T_f$.

Control appropriate (slower) sampling period: $T_s$
(typically $T_s = MT_f$ for integer $M > 1$).