A Short Review of Prerequisites for Regelsysteme 2

These are some short notes which should help in giving an overview of the knowledge required in order to be able to follow the course Regelsysteme 2 without problems. They are mostly covered by any basic control course, in particular Regelsysteme 1 from the previous semester. While all the essential topics are listed, not all of them can be covered in sufficient depth due to time limitations. These notes are thus not suitable for studying the material—their purpose is to provide an overview of prerequisites, rather to teach the material in a single exercise class. I would like to remind you of what you already know about control, and provide some outlook and motivation for what will come in this class.

If you feel unfamiliar with any of the topics discussed here, you are advised to go back to your class notes from the previous semester, or some of the abundant literature, in order to review those. Appropriate references will be provided for each of the following sections.

1 Why Feedback Control?

Figure 1.1 depicts a standard control system for reference tracking. Here the variables indicate the following:

- \( G(s) \): plant model
- \( K(s) \): feedback controller (using \( e(s) \))
- \( F_r(s) \): feedforward controller on the reference \( r(s) \)
- \( F_d(s) \): feedforward controller on the disturbance \( d(s) \)
- \( r(s) \): reference inputs (commands, setpoints)
- \( e(s) \): control error
- \( u(s) \): plant input
- \( d(s) \): process disturbance
- \( n(s) \): measurement noise
- \( y(s) \): plant output

![Figure 1.1: Control System with Feedback and Feedforward](image)

The most common purpose of designing a control system is to achieve the tracking of some input signal \( r(t) \) by the system output \( y(t) \). In particular, the following attributes are often considered:

- low tracking offset (in steady-state),
- fast reaction time,
- little overshoot.
We are particularly interested in feedback controllers, because they have proven to work well in many applications. Feedforward controllers suffer from a number of drawbacks that shall be demonstrated by a small example. Consider the control system depicted in Figure 1.1 and suppose it is operated without feedback, that is by setting $K(s) := 0$. We choose the feedforward controller on the reference signal $F_r(s) := G^{-1}(s)$ and on the disturbance $F_d(s) := G^{-1}(s)$ in order to achieve perfect tracking: now $y(s) = r(s)$. But there are certain problems with this approach:

- the plant model $G(s)$ contains uncertainties,
- the process disturbances is usually unknown,
- $G^{-1}(s)$ may be unrealizable or unstable.

This provides a small indication of why feedback control has proved to be a powerful tool, although its tracking performance is subject to inherent limitations set by the system at hand. This topic will be addressed further in this class. Feedforward control is not a viable alternative in most cases, mainly for the reasons above. Nonetheless, it may sometimes be combined appropriately with feedback control in order to improve the control performance, for example in inverse-based control.

The following sections review a variety of topics from the basic theory of feedback control, as taught in most basic control courses, e.g. from Regelsysteme 1. It is important that you are familiar with them in order to properly follow this class. This overview does not intend to give the full details on these topics, so if you would feel unfamiliar with any of the topics below, please review them based on one of a whole variety of excellent text books on control theory, e.g. [1], [2], [3], [4], [5], [6], [7].

2 State Space Representation

2.1 Linear and Nonlinear Models

The most general form of a dynamic model to be considered here is in continuous time, nonlinear state space model with finitely many states:

\begin{align}
\dot{x}(t) &= f(x(t), u(t)) , \\
y(t) &= g(x(t), u(t)) .
\end{align}

It consists of a state equation (2.1a) describing the dynamics of the state $x(t) \in \mathbb{R}^n$ given the external input $u(t) \in \mathbb{R}^m$ according to some function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and the output equation (2.1b) producing the output $y(t) \in \mathbb{R}^p$ from the state $x(t)$ and the input $u(t)$ via some function $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$.

While there is also a large body of theory on nonlinear control, in both courses Regelsysteme 1 and 2 we are concerned with the theory of linear control systems only, which take the form of a linear state space model:

\begin{align}
\dot{x}(t) &= Ax(t) + Bu(t) , \\
y(t) &= Cx(t) + Du(t) .
\end{align}

Here $A \in \mathbb{R}^{n \times n}$ is the system matrix, $B \in \mathbb{R}^{n \times m}$ is the input matrix, $C \in \mathbb{R}^{p \times n}$ is the output matrix, and $D \in \mathbb{R}^{p \times m}$ is the feedthrough matrix. A block diagram of (2.2) is drawn in Figure 2.1.

The essence of the state space models (2.1) and (2.2) is that there are some dynamics in the states $x(t)$ that are dictated by the system to be considered, but they can be influenced by an appropriate selection of inputs $u(t)$ given to the system. An additional complication results from the fact that not full information about the states $x(t)$ may be given, as only some of them may be measured as part of the outputs $y(t)$. 
Many control systems arising from mechanical or electrical systems are linear and hence come in the form of (2.2). Control systems that behave according to nonlinear dynamics (2.1) can often be linearized around an operating point $x_0 \in \mathbb{R}^n$, that is a state around which the system usually operates. In this case, a local approximation of the form (2.2) can often be obtained by means of linearization, based on which a controller can then be designed.

**Prerequisites:** Deriving linear state space models for basic mechanical and electrical systems, and from nonlinear state space models by linearization, e.g. [1, Chapter2], [4, Chapter2], [2, Chapter4], [6, Chapter2].

### 2.2 Transfer Functions

Many linear controller design procedures take place in the *frequency domain*. Switching from the time domain point-of-view to that of the frequency domain means shifting from the input-output behavior $u(t)$ to $y(t)$ in terms of time $t \in \mathbb{R}_{0+}$ to the input-output behavior of $u(s)$ to $y(s)$ in terms of frequency $s \in \mathbb{C}$. The *Laplace transform* (with complex frequency $s \in \mathbb{C}$) and the *Fourier transform* (with the real frequency $\omega \in \mathbb{R}_{0+}$), together with their inverses, serve as translations between the two domains.

Roughly speaking, in the transformed domain the input-output behavior of the plant is described in terms of its reaction to sinusoidal input signals $u(t) = \cos(\omega t)$ of varying frequency $\omega \in \mathbb{R}_{0+}$. Since system (2.2) is linear, the output is known to be again be a sinusoid of the same frequency, i.e.

$$y(t) = M(\omega) \cos(\omega t + \phi(\omega)),$$

where $M$ is an *amplification* and $\phi$ is a *phase shift*. Both of them depend on the choice of the frequency $\omega$, and their behavior over the entire frequency range $M(\omega)$ and $\phi(\omega)$ completely characterizes the linear plant, which can thus be represented just as a complex number over the entire frequency range:

$$G(j\omega) := M(\omega)e^{j\phi(\omega)} \quad \forall \omega \in \mathbb{R}_{0+}.$$  

(2.3)

The frequency-domain viewpoint is illustrated in Figure 2.2, contrasting to the time-domain viewpoint in Figure 2.1.

$$u(t) = \cos(\omega t) \quad \rightarrow \quad G(j\omega) \quad \rightarrow \quad y(t) = M(\omega) \cos(\omega t + \phi(\omega))$$

Figure 2.2: Frequency Response of a Linear Model $G(j\omega)$

More precisely, the Laplace transform of the linear system (2.2), also known as its *transfer function*, can be obtained by the fundamental formula

$$G(s) = C(sI - A)^{-1}B + D.$$  

(2.4)
The transfer function in (2.4) is a rational function in \( s \),
\[
G(s) = \frac{n(s)}{d(s)} = \frac{a_n s^n + \ldots + a_1 s + a_0}{b_n s^n + \ldots + b_1 s + b_0},
\]
(2.5)
where the degree of the numerator polynomial \( n(s) \) is at most equal to that of the denominator polynomial \( d(s) \). The roots of \( n(s) \) are called the zeros of the system, and the roots of \( d(s) \) are called its poles.

Given the frequency-domain representation of the input \( u(s) \), the frequency-domain output \( y(s) \) can simply be obtained by means of multiplication:
\[
y(s) = G(s)u(s).
\]
Notice that we will switch back and forth between the Laplace transform and the Fourier transform by setting \( s = j\omega \) in signals as well as transfer functions.

**Prerequisites:** Use of the Laplace transform and Fourier transform for solving linear differential equations, and application to linear control systems e.g. [1, Appendix A], [4, Appendix B], [2, Chapter 6], [6, Chapter 2].

### 2.3 Closed-Loop Transfer Functions

In the frequency domain, real systems display a low-pass filter behavior, i.e. low-frequency inputs are associated with a high amplification whereas high-frequency inputs tend to have a low amplification as they are associated with a strong damping. This is expressed, in mathematical terms, by the transfer functions being ‘strictly proper’ (or at least ‘semi-proper’).

**Definition 2.1.** A linear system represented by the transfer function \( G(s) \) is **strictly proper** if \( G(j\omega) \to 0 \) as \( \omega \to \infty \). \( G(s) \) is **semi-proper** if \( G(j\omega) \to C \neq 0 \) as \( \omega \to \infty \). If a linear system is either strictly proper or semi-proper, then it is also called **proper**; if it is neither, then it is **improper**.

In terms of its rational transfer function (2.5), a system is strictly proper if and only if the degree of its denominator polynomial is strictly larger than that of its numerator polynomial, and it is semi-proper if and only if the two degrees are identical.

In the frequency domain, the solution to the system depicted in Figure 1.1 (where \( F_r(s) = 0 \) and \( F_d(s) \)) follows as
\[
y(s) = \frac{G(s)K(s)}{1 + G(s)K(s)} r(s) + \frac{1}{1 + G(s)K(s)} d(s) - \frac{G(s)K(s)}{1 + G(s)K(s)} v(s),
\]
(2.6)
because of the superposition principle for linear systems. Based on the notation in [7], the abbreviation of \( L(s) := G(s)K(s) \) for the loop transfer function will sometimes be used.

The closed-loop transfer functions \( S(s) \) and \( T(s) \) are called the **sensitivity function** and **complementary sensitivity function**. From the formulas in (2.6) it is easy to see that for strictly proper loop transfer functions \( L(s) \) the following must hold:
\[
S(j\omega) \to 1 \quad \text{and} \quad T(j\omega) \to 0 \quad \text{as} \quad \omega \to \infty.
\]

So the complementary sensitivity function, expressing the plant’s behavior from the reference to the output (and the noise to the output), also displays a low-pass filter behavior. In contrast, the sensitivity function, expressing the plant’s behavior from the disturbance to the output, displays a high-pass filter behavior. This, apparently, may lead to a poor damping of disturbances at high frequencies.

**Prerequisites:** Block diagrams and computing transfer functions from block diagrams. The superposition principle for linear systems, e.g. [1, Section 3.2], [2, Chapter 3], [6, Chapter 5].
2.4 Stability

The most important requirement in feedback control design is stability. The lack of stability of a control system can have horrible physical consequences, and may lead to a destruction of the system.

**Definition 2.2.** A control system is said to be (internally) stable if for any initial state of the system and in the absence of any control inputs $u(t) = 0$, the state trajectory converges to the origin.

This definition can be applied to linear systems (2.2) as well as in the nonlinear case (2.1), see [1, p. 130]. For linear systems, however, stability can be conveniently tested by examination of the characteristic values of the system. In the form of a state space representation (2.2) they correspond to the eigenvalues of the system matrix $A$. Almost equivalently, in the form of a transfer function (2.5) they correspond to the poles—unless in cases where a pole-zero cancellation takes place. The case of a pole-zero cancellation implies that there is some mode of the system which is not reflected in its input-output behavior, also called a hidden mode. For the stability of the system it is essential, however, that any hidden mode is also stable, see [1, p. 131].

**Theorem 2.3.** A linear system is (internally) stable if and only if all of its characteristic values are in the open left-half plane (LHP) of the complex plane.

Sometimes weaker notions of stability are defined, such as bounded-input bounded-output (BIBO) stability, but they are not considered in this class. One may speak of marginally stable systems if some characteristic values of the system lie on the imaginary axis and are not repeated [1, p. 131].

**Prerequisites:** Notion of stability of an open-loop and a closed-loop system, e.g. [1, Section 3.7], [4, Section 3.4], [2, Chapter 8], [6, Section 6.1].

2.5 Controllability and Observability

**Definition 2.4.** A linear system (2.2) is said to be controllable if there exists an input signal $u(t)$ which takes any state $x(t_0)$ to any desired value $x(t_0 + \Delta t)$ in some finite time $\Delta t$.

In the case of linear systems, both the final state $x(t_0 + \Delta t)$ and the time interval $\Delta t$ turn out not to matter for the criterion—that is, if a linear system can be driven for instance to the origin in some finite time $\Delta t$, then it can be driven to any arbitrary state in any arbitrary finite time.

In essence, the only thing that is needed for a system to be controllable is that every mode of the system can somehow be influenced by the control input $u(t)$. It turns out that this holds true if and only if the controllability matrix

$$P := \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}$$

has full row rank $n$.

Observability can be seen as a dual property of controllability for linear systems [1, p. 851]. So all of that was said above applies analogously to the following definition.

**Definition 2.5.** A linear system (2.2) is said to be observable if the state $x(t)$ can be computed by measuring the input signal $u(t)$ and the output signal $y(t)$ over a finite time interval $\Delta t$. 
In essence, the only thing that is needed for a system to be observable is that every mode of the system somehow influences the output $y(t)$. It turns out that this holds true if and only if the observability matrix

$$Q := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has full column rank $n$.

Both controllability and observability depend on the state space realization of a system (2.2), and not on the input-output behavior described by the corresponding transfer function. In fact, a pole-zero cancellation (see Section 2.4) occurs exactly if a system is not both controllable and observable, in which case it contains hidden modes which do not show up in its input-output behavior.

**Prerequisites:** Controllability and observability of linear systems, e.g. [1, Appendix D], [4, Chapter 5], [5, Chapter 7], [3, Chapter 3], [6, Chapter 12].

### 2.6 Canonical Realizations and Similarity Transforms

Let $G(s)$ be any transfer function which is strictly proper, according to the definition in Section 2.3. Then it has the following state space realizations:

- controllable canonical representation,
- observable canonical representation,
- Jordan normal form.

**Prerequisites:** Various state space representations and their properties, invariance of controllability and observability under similarity transformations, e.g. [1, Appendix D], [4, Chapter 5], [5, Chapter 7], [3, Chapter 3], [6, Chapter 12].

### 3 Frequency Response Characterization

#### 3.1 Bode Diagrams

Bode diagrams describe the behavior of a transfer function $G(j\omega)$ over the whole range of frequencies $\omega \in \mathbb{R}_{0+}$, also called the frequency response of the system. According to (2.3), for some fixed frequency $\omega$ the transfer function $G(j\omega)$ represents only a complex number and it can thus be expressed in polar form as

$$G(j\omega) = M(\omega) \cdot \exp\left[j \phi(\omega)\right]. \quad (3.1)$$

The magnitude $M(\omega)$ and the phase $\phi(\omega)$ are real functions of the frequency and can be plotted over $\omega$, giving the magnitude plot and the phase plot of $G(s)$, respectively. Both plots combined constitute the Bode diagram.

The Bode diagram has some important special features for its convenient use, and they need to be born in mind whenever working with Bode diagrams.

- The abscissas of both plots show $\omega \in \mathbb{R}_{0+}$ in logarithmic scale:
The phase angles of two multiplying transfer functions can simply be added:

\[ G(j\omega) = G_1(j\omega)G_2(j\omega) \implies \phi(\omega) = \phi_1(\omega) + \phi_2(\omega). \]

This follows from the basic laws of complex numbers.

The magnitude plot does not have \( M(\omega) \) on the ordinate, but \( 20 \log[M(\omega)] \), where \( \log[\cdot] \) denotes the 10-logarithm. It is measured in the ‘decibel’ units (‘dB’). This has the advantage that the magnitudes of two multiplying transfer functions can be also added instead of having to be multiplied:

\[ G(j\omega) = G_1(j\omega)G_2(j\omega) \implies M(\omega) = M_1(\omega)M_2(\omega) \implies 20 \log[M(\omega)] = 20 \log[M_1(\omega)] + 20 \log[M_2(\omega)]. \]

This follows from the basic laws of calculus for logarithms.

The addition property for the magnitude and the phase has a huge advantage for sketching the Bode diagram of rational transfer functions \( G(s) \). Indeed, the rational function of (2.5) can be factored into its zeros and poles,

\[ G(s) = \frac{n(s)}{d(s)} = \frac{(s - z_1)(s - z_2)...}{(s - p_1)(s - p_2)...}, \quad (3.3) \]

where \( z_i \) denote the real zeros and \( p_i \) denote the real poles. On top of the real poles, many systems also contain a pair of complex conjugate poles, that is \( d(s) \) also has a factor of the general form \((s/\omega_n)^2 + 2\zeta(s/\omega_n) + 1\). The addition property for the magnitude now implies that the Bode diagram of the complicated function \( G(s) \) can simply be constructed by adding together all the elementary Bode diagrams of its factors. Hence it can be approximated easily by knowing approximations to the elementary Bode diagrams of the standard first-order and second-order elements, which are shown in Figure 3.1.
\[ G(s) = s + a \]

\[ G(s) = \frac{1}{s + a} \]

\[ G(s) = -s + a \]

\[ G(s) = \frac{1}{s + a} \]

Figure 3.1: Basic Bode Diagram Approximations

**Prerequisites:** Interpretation of Bode diagrams and sketching approximative Bode diagrams for given transfer functions, e.g. [1, Section 6.1], [2, Section 6.3], [6, Section 10.2], [7, Section 2.1].
3.2 Nyquist Diagrams

Another way to represent the frequency response of \( G(j\omega) \) is the Nyquist diagram. The Nyquist diagram itself is closed contour in the complex plane, which arises as the image of the Nyquist D-contour (depicted in Figure 3.2) under \( G(s) \) viewed as a map \( G : \mathbb{C} \to \mathbb{C} \). This works as \( G(s) \) is a complex number for any complex number \( s \in \mathbb{C} \).

![Nyquist D-contour](image)

Figure 3.2: Nyquist D-contour

Nyquist diagrams are always symmetric with respect to the real axis. For strictly proper systems, the half-circle at infinity shrinks to the origin; then the plot consists of two symmetric branches \( \omega \in (-\infty, 0) \) and \( \omega \in (0, +\infty) \), where the latter branch is closely related to the Bode diagram as follows:

- The Nyquist branch is a curve parameterized by \( \omega \in \mathbb{R}_{0^+} \), i.e., for each \( \omega \) there is a point on the Nyquist curve.
- The distance of this point to the origin equals to \( M(\omega) \).
- The angle of this point with the positive real axis equals to \( \phi(\omega) \).

Prerequisites: Interpretation of Nyquist diagrams and the connection between Bode and Nyquist diagrams, e.g. [1, Section 6.3], [2, Section 8.5], [6, Section 10.4], [7, Section 2.4.3]. You will not be asked to draw an approximative Nyquist diagram in this course, but if you are able to sketch it for a given transfer functions, that is even better.

4 Nyquist and Bode Stability

4.1 The Argument Principle

The Nyquist stability criterion is based on the argument principle, stated in abstract form below.

Lemma 4.1. Let \( L(s) \) be a meromorphic function defined on the complex plane \( \mathbb{C} \) or, in particular, a rational function. Let \( C \) be simple closed contour with clockwise direction in \( \mathbb{C} \) such that no zeros or poles
of \( L(s) \) lie on \( C \). Let \( Z \) and \( P \) be the number of zeros and poles of \( L(s) \) inside of \( C \). Then the image contour \( G(C) \) encircles the origin \((Z - P)\) times in clockwise direction.

In order to test for stability, one has to obtain the number poles in the closed right-half plane. For this reason, the Nyquist stability criterion (usually) applies the Nyquist D-contour from Figure 3.2, where the radius \( R \to \infty \) and therefore it covers exactly the entire right-half plane. Special care has to be taken in the case when poles lie on the imaginary axis, in which case small modifications have to be made to the contour.

### 4.2 Nyquist Stability Criterion

The Nyquist stability criterion is a clever exploitation of the argument principle, in order to assess the stability of the standard feedback loop in Figure 4.1.

![Figure 4.1: Standard Feedback Loop](image)

First, one can lump together the plant and the controller into the loop transfer function \( L(s) = G(s)K(s) \), and express the closed-loop transfer function \( T(s) \) in terms of the numerator polynomial \( n(s) \) and denominator polynomial \( d(s) \),

\[
L(s) = \frac{n(s)}{d(s)} \implies T(s) = \frac{n(s)}{d(s) + n(s)}.
\]

Next, consider the ‘artificial’ transfer function \( H(s) \) defined by

\[
H(s) := 1 + L(s) \implies H(s) := \frac{d(s) + n(s)}{d(s)} \quad (4.1)
\]

and notice that \( H(s) \) has the closed-loop poles in the numerator and the open-loop poles in the denominator. Hence by the argument principle, the number of clockwise encirclements of the origin by the Nyquist contour of \( H(s) \) equals to the number of RHP poles of \( T(s) \) minus the number of RHP poles of \( L(s) \).

But according to (4.1), the Nyquist contour of \( H(s) \) is nothing but the Nyquist contour of \( L(s) \) shifted by +1 to the right. Thus we may instead consider the encirclements of the point \(-1 + 0j\) by the Nyquist contour of \( L(s) \) and obtain the following result.

**Theorem 4.2 (Nyquist Stability Criterion).** Define the variables \( P, N, Z \) as follows:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>( P )</td>
<td>number of (open-loop) poles of ( L(s) ) in the open right-half plane,</td>
</tr>
<tr>
<td>( N )</td>
<td>number of encirclements of the point (-1) by the Nyquist diagram of ( L(s) ) (counting clockwise as positive, counter-clockwise as negative),</td>
</tr>
<tr>
<td>( Z )</td>
<td>number of (closed-loop) poles of ( T(s) ) in the open right-half plane.</td>
</tr>
</tbody>
</table>

The number of closed-loop poles in the RHP relates to the number of open-loop poles in the RHP and the Nyquist diagram as follows:

\[
Z = N + P.
\]

Therefore closed-loop stability requires \( N = -P \), and in cases where the open loop is stable, \( N = 0 \).
**Prerequisites:** Theory and application of the Nyquist stability criterion, based on the argument principle e.g. [1, Section 6.3], [2, Section 8.5], [6, Section 10.5].

### 4.3 Gain Margin and Phase Margin

Gain margin and phase margin measure the ‘proximity’ for a loop transfer function $L(s)$ to its closed-loop instability, i.e. the instability of the standard feedback loop of Figure 4.1. They also serve as an indicator of robustness (in terms of stability) against parameter uncertainties and disturbances. More details on this will be discussed in this class.

**Definition 4.3.** The *gain margin* (GM) of the system $L(s)$ is the scalar factor (‘gain’) by which $L(s)$ can be multiplied without the closed-loop system becoming unstable.

The *phase margin* (PM) of the system $L(s)$ is the maximum phase lag $\theta$ which can be added to $L(s)$ (i.e. multiplied with phase lag element $e^{-\theta s}$) without the closed-loop system becoming unstable.

From the derivation it is clear that GM and PM can be found in the Nyquist diagram as illustrated in Figure 4.2 below.

![Figure 4.2: GM and PM in Nyquist Diagram](image)

**Prerequisites:** Stability margins in the Nyquist diagram, e.g. [1, Section 6.4], [2, Section 8.5], [6, Section 10.6], [7, Section 2.4.3].

### 4.4 Bode Stability Criterion

Suppose that $L(s)$ is stable and its magnitude and phase are monotonely decreasing with the frequency $\omega$. Then the gain margin and phase margin can also be found in the Bode diagram as illustrated in Figure 4.3. This also implies the following stability criterion.
Theorem 4.4 (Bode Stability Criterion). Suppose that $L(s)$ is stable and its magnitude and phase are monotonely decreasing with the frequency $\omega$. Then the standard feedback loop of Figure 4.1 is stable if and only if

$$M(\omega) < 1 \quad \text{where} \quad \phi(\omega) = -180^\circ.$$ 

![Bode Diagram](image)

Figure 4.3: GM and PM in Bode Diagram

Prerequisites: Stability margins and stability in the Bode diagram, e.g. [1, Section 6.4], [2, Section 8.5.5], [6, Section 10.7], [7, Section 2.6].

References


