A Short Review of Prerequisites for Regelsysteme 2

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Motivation

- Review of basic concepts required to follow Regelsysteme 2 course.
- Has been taught in Regelsysteme 1 course. Can also be found in any introductory control book.
- If anything sounds new (or scary) to you, you are encouraged to read up on it or talk to us. - Suitable references are provided in the notes.
Outline

Why Feedback Control

State Space Representation

Frequency Response Characterization

Nyquist and Bode Stability
Control Loop

- $u(s)$ plant input
- $y(s)$ plant output
- $d(s)$ process disturbance
- $G(s)$ plant model
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$u(s)$ plant input
$y(s)$ plant output
$d(s)$ process disturbance
$r(s)$ reference input
$n(s)$ measurement noise

$G(s)$ plant model
$K(s)$ feedback controller
$F_r(s)$ feedforward controller
$F_d(s)$ feedforward controller
Control Loop

- $u(s)$ plant input
- $y(s)$ plant output
- $d(s)$ process disturbance
- $r(s)$ reference input
- $n(s)$ measurement noise
- $G(s)$ plant model
- $K(s)$ feedback controller
- $F_r(s)$ feedforward controller
- $F_d(s)$ feedforward controller
Control Loop

- $u(s)$: plant input
- $y(s)$: plant output
- $d(s)$: process disturbance
- $r(s)$: reference input
- $n(s)$: measurement noise
- $G(s)$: plant model
- $K(s)$: feedback controller
- $F_r(s)$: feedforward controller
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- $n(s)$: measurement noise
- $G(s)$: plant model
- $K(s)$: feedback controller
- $Fr(s)$: feedforward controller
- $Fd(s)$: feedforward controller
Control Methods

Control Objectives

▶ stabilization
▶ low steady-state tracking offset
▶ fast reaction time
▶ little overshoot

Design Tools

▶ feedback controller $K(s)$
▶ feedforward controllers $F_r(s)$, $F_d(s)$
Feedforward Control

- No feedback control, i.e. \( K(s) = 0 \)
- Invert the plant \( G(s) \)
- i.e. \( F_r(s) \triangleq G^{-1}(s) \) and \( F_d(s) \triangleq G^{-1}(s) \)
Feedforward Control

- no feedback control, i.e. $K(s) = 0$
- invert the plant $G(s)$
- i.e. $F_r(s) \triangleq G^{-1}(s)$ and $F_d(s) \triangleq G^{-1}(s)$

$$y(s) = G^{-1}(s)G(s)r(s) - G^{-1}(s)G(s)d(s) + d(s) = r(s)$$
Why Feedback Control

Feedforward controllers (with $K(s) = 0$) typically choose $F_r = G^{-1}$, $F_d = G^{-1}$. However, such an approach is subject to certain limitations:

- signal uncertainties, unknown disturbance $d(s)$
- $G^{-1}$ may be unstable or unrealizable
- model uncertainties in plant model $G(s)$
Why Feedback Control

Feedforward controllers (with $K(s) = 0$) typically choose $F_r = G^{-1}$, $F_d = G^{-1}$. However, such an approach is subject to certain limitations:

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- $G^{-1}$ may be unstable or unrealizable
- model uncertainties in plant model $G(s)$

Therefore, most of control theory deals with feedback controllers. They have proven to work well in many applications since they are able to deal with the points above.
Why Feedback Control

- model uncertainties in plant model $G(s)$

Consider the two plants $G_1(s) = \frac{1}{s + 1}$ and $G_2(s) = \frac{1}{s - 1}$
Why Feedback Control

- model uncertainties in plant model $G(s)$

We apply $K = 100$ as a feedback controller to both
State Space Representation

General model:

\[ \dot{x}(t) = f(x(t), u(t)), \]
\[ y(t) = g(x(t), u(t)). \]
State Space Representation

General model:

\[
\dot{x}(t) = f(x(t), u(t)), \\
y(t) = g(x(t), u(t)).
\]

Linear (time-invariant) model:

\[
\dot{x}(t) = Ax(t) + Bu(t), \\
y(t) = Cx(t) + Du(t).
\]

- \(A \in \mathbb{R}^{n \times n}\) is the system matrix
- \(B \in \mathbb{R}^{n \times m}\) is the input matrix
- \(C \in \mathbb{R}^{p \times n}\) is the output matrix
- \(D \in \mathbb{R}^{p \times m}\) is the feedthrough matrix
Transfer Function and Frequency Response

\[ u(t) = \cos(\omega t) \rightarrow G(j\omega) \rightarrow y(t) = M(\omega) \cos(\omega t + \phi(\omega)) \]

Captured by **Laplace Transform**:

- \( y(s) = G(s)u(s) \)
- \( G(s) = C(sI - A)^{-1}B + D \)
- \( G(s) = \frac{n(s)}{d(s)} = \frac{a_n s^n + \ldots + a_1 s + a_0}{b_n s^n + \ldots + b_1 s + b_0} \)
Transfer Function and Frequency Response

\[ u(t) = \cos(\omega t) \quad \Rightarrow \quad G(j\omega) \quad \Rightarrow \quad y(t) = M(\omega) \cos(\omega t + \phi(\omega)) \]

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and **Fourier Transform** by setting \( s = j\omega \):

- \[ G(j\omega) = M(\omega)e^{j\phi(\omega)} \]

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Closed Loop Transfer Function

By superposition principle it follows:

\[
y(s) = \frac{G(s)K(s)}{1 + G(s)K(s)} r(s) + \frac{1}{1 + G(s)K(s)} d(s) - \frac{G(s)K(s)}{1 + G(s)K(s)} n(s)
\]
The (Complementary) Sensitivity Function

- loop transfer function: \( L(s) \triangleq G(s)K(s) \)
- sensitivity function: \( S(s) \triangleq \frac{1}{1 + L(s)} \)
- complementary sensitivity function: \( T(s) \triangleq \frac{L(s)}{1 + L(s)} \)
The (Complementary) Sensitivity Function

- **loop transfer function**: \( L(s) \triangleq G(s)K(s) \)
- **sensitivity function**: \( S(s) \triangleq \frac{1}{1 + L(s)} \)
- **complementary sensitivity function**: \( T(s) \triangleq \frac{L(s)}{1 + L(s)} \)

Hence, we can write

\[
y(s) = T(s)r(s) + S(s)d(s) - T(s)n(s)
\]
Stability

Definition
A system is (internally) stable if for any initial state and $u(t) = 0$ the state trajectory converges to the origin.
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Theorem
A linear system is (internally) stable if and only if all of its characteristic values are in the open left half plane (oLHP) of the complex plane.

- In the state space representation, the characteristic values correspond to the eigenvalues of \( A \).
- In the transfer function form, they correspond to the poles – unless there is a pole-zero cancellation.
Controllability

Definition
A system is said to be **controllable** if there exists an input signal \( u(t) \) which takes any initial state \( x(t_0) \) to any desired value \( x(t_0 + \Delta t) \) in some finite time \( \Delta t \).
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Theorem
A linear system is controllable if and only if the controllability matrix

\[
C \triangleq \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}
\]

has full (row) rank.

- Intuitively, this means that every mode of the system can somehow be influenced by the control input \( u(t) \).
Observability

Definition
A system is said to be **observable** if the state $x(t)$ can be computed by measuring the input signal $u(t)$ and the output signal $y(t)$ over a finite time interval $\Delta t$. 

Theorem
A linear system is observable if and only if the observability matrix $Q \equiv \begin{bmatrix} C & CA & \cdots & CA^{n-1} \end{bmatrix}$ has full (column) rank.

▶ Intuitively, this means that every mode of the system somehow influences the output $y(t)$. 
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A linear system is observable if and only if the observability matrix

$$Q \triangleq \begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{bmatrix}$$

has full (column) rank.

- Intuitively, this means that every mode of the system somehow influences the output $y(t)$. 
A transfer function $G(s)$ has the following canonical state space representations:

- controllable canonical representation
- observable canonical representation
- modal representation, or more generally, Jordan normal representation

Each canonical form can be transformed into another by similarity transforms if it exists.
Much of (linear) controller design takes place in the *frequency domain*, e.g. gain margin, phase margin. In particular, we have two tools at hand:

- Bode diagram
- Nyquist diagram
They describe the behavior of the frequency response \( G(j\omega) \) for \( \omega \geq 0 \). Recall that \( G(j\omega) \) can be written as
\[
G(j\omega) = M(\omega) \cdot e^{j\phi(\omega)},
\]
where
- \( M(\omega) \) is the **magnitude** of \( G(j\omega) \)
- \( \phi(\omega) \) is the **phase** of \( G(j\omega) \)

If \( M(\omega) \) and \( \phi(\omega) \) are plotted over \( \omega \), we obtain the **Bode diagram**.
Bode Diagrams

- The frequency axes are in logarithmic scale.
- Addition rule for phase:

\[
G(j\omega) = G_1(j\omega)G_2(j\omega)
\]

\[
\implies G(j\omega) = M_1(\omega)M_2(\omega) \cdot \exp\left[j \left(\phi_1(\omega) + \phi_2(\omega)\right)\right].
\]

- Logarithmic scale on magnitude measured in **decibel** (dB):

\[
M(\omega) = M_1(\omega)M_2(\omega)
\]

\[
\implies 20 \log[M(\omega)] = 20 \log[M_1(\omega)] + 20 \log[M_2(\omega)].
\]
Bode Diagrams

The above rules allow us to view complicated $G(s)$ as a system of simpler ones by factorization

$$G(s) = \frac{n(s)}{d(s)} = \frac{(s - z_1)(s - z_2)}{(s - p_1)(s - p_2)}...$$

- Addition properties say that we can add the phases and magnitudes of all the elementary Bode plots together.
Differentiator and Integrator

\[ G(s) = s \]

\[ 20 \log [M(\omega)] \]

\[ \phi(\omega) \]

\[ \omega \text{ [rad/s]} \]

\[ +20 \text{ dB/decade} \]

\[ \omega \text{ [rad/s]} \]

\[ -90^\circ \]

\[ \omega \text{ [rad/s]} \]

\[ \phi(\omega) \]

\[ \omega \text{ [rad/s]} \]

\[ -90^\circ \]

\[ G(s) = \frac{1}{s} \]

\[ 20 \log [M(\omega)] \]

\[ \phi(\omega) \]

\[ \omega \text{ [rad/s]} \]

\[ -20 \text{ dB/decade} \]

\[ \omega \text{ [rad/s]} \]

\[ \phi(\omega) \]

\[ \omega \text{ [rad/s]} \]

\[ -90^\circ \]
Pole/Zero in open Left Half Plane

\[ G(s) = s + a \]

\[ G(s) = \frac{1}{s + a} \]

\[ 20 \log [M(\omega)] \]

\[ +20 \text{ dB/decade} \]

\[ -20 \text{ dB/decade} \]

\[ \phi(\omega) \]

\[ +45^\circ/\text{decade} \]

\[ -45^\circ/\text{decade} \]
Pole/Zero in open Right Half Plane

\[ G(s) = -s + a \]

\[ \frac{1}{-s + a} \]

\[ 20 \log[M(\omega)] \]

\[ +20 \text{ dB/decade} \]

\[ -20 \text{ dB/decade} \]

\[ \phi(\omega) \]

\[ -45^\circ/\text{decade} \]

\[ +45^\circ/\text{decade} \]
Nyquist Diagrams

The Nyquist diagram is a closed contour in the complex plane, which arises as the image of the Nyquist D-contour under the complex mapping $G : \mathbb{C} \rightarrow \mathbb{C}$.
Some properties of the Nyquist diagram:

- Nyquist diagrams are symmetric w.r.t. the real axis.
- For strictly proper systems, the half-circle at infinity shrinks to the origin, and we only need to consider the image under the $j\omega$ axis.
- The branch parametrized by the $j\omega$ axis is closely related to the Bode diagram in the following sense:
  1. The distance of this point to the origin equals $M(\omega)$.
  2. The angle of this point with the positive real axis equals to $\phi(\omega)$. 
Nyquist and Bode Stability

Standard feedback loop:

\[ r(s) \rightarrow K(s) \rightarrow u(s) \rightarrow G(s) \rightarrow y(s) \]

We want to assess the stability of the plant \( G(s) \) with the controller \( K(s) \) in closed loop in frequency domain by looking at \( L(s) = G(s)K(s) \). For this, we have two tools at hand:

- Bode stability criterion
- Nyquist stability criterion
Gain Margin and Phase Margin

*Gain margin (GM)* and *phase margin (PM)* measure the ‘proximity’ for $L(s)$ to its closed-loop instability.
Gain Margin and Phase Margin

*Gain margin (GM)* and *phase margin (PM)* measure the ‘proximity’ for \( L(s) \) to its closed-loop instability.

**Definition**
The **gain margin (GM)** of the system \( L(s) \) is the scalar factor (‘gain’) by which \( L(s) \) can be multiplied without the closed-loop system becoming unstable.

**Definition**
The **phase margin (PM)** of the system \( L(s) \) is the maximum phase lag \( \theta \) which can be added to \( L(s) \) (i.e. multiplied with phase lag element \( e^{-\theta s} \)) without the closed-loop system becoming unstable.
Gain Margin and Phase Margin

*Gain margin (GM)* and *phase margin (PM)* can serve as an indicator of robustness to its instability against parameter uncertainties and disturbances.
Theorem

Suppose that \( L(s) = G(s)K(s) \) is stable, and its magnitude and phase are monotonously decreasing with frequency \( \omega \). Then the standard feedback loop is stable if and only if

\[
M(\omega_{180}) < 1 \quad \text{where} \quad \phi(\omega_{180}) = -180^\circ.
\]

Equivalently, it is stable if and only if

\[
\phi(\omega_c) > -180^\circ \quad \text{where} \quad M(\omega_c) = 1.
\]
If $L(s)$ is stable with decreasing magnitude and phase
$\implies$ system is stable according to Bode criterion!
Nyquist Stability Criterion

The Nyquist stability criterion is a more general method for assessing the closed loop stability.

Application:
Define the variables $P$, $N$, $Z$ as follows:

- $P$: number of (open-loop) poles of $L(s)$ in the open right-half plane,
- $N$: number of encirclements of the point $-1$ by the Nyquist diagram of $L(s)$ (counting clockwise as positive, counter-clockwise as negative),
- $Z$: number of (closed-loop) poles of $T(s)$ in the open right-half plane.
Nyquist Stability Theorem

Theorem (Nyquist Stability Criterion)

The number of closed-loop poles in the RHP relates to the number of open-loop poles in the RHP and the Nyquist diagram as follows:

\[ Z = N + P. \]

Therefore closed-loop stability requires \( N = -P \), and in cases where the open loop is stable, \( N = 0 \).
Questions?