CONTROL SYSTEMS II (REGELSYSTEME II)

SOLUTIONS TO PROBLEM SET 3

Exercise 1  Interpolation Constraints

The loop transfer function $L(s)$ can be written as follows,

$$
L(s) = \frac{N(s)}{D(s)}
$$

(1)

then the sensitivity function $S(s)$ and the complementary sensitivity function $T(s)$ will be,

$$
S(s) = \frac{1}{1 + L(s)} = \frac{D(s)}{D(s) + N(s)}
$$

(2a)

$$
T(s) = \frac{L(s)}{D(s) + L(s)} = \frac{N(s)}{D(s) + N(s)}
$$

(2b)

Assuming that $G(s)$ has a RHP-zero at $z$, then internal stability requirement implies that $L = GK$ will also have a RHP-zero at $z$. Therefore, equation (2) implies that,

$$
N(z) = 0 \Rightarrow T(z) = 0, \quad S(z) = 1
$$

Similarly, if $G(s)$ has a RHP-pole at $p$, then internal stability requirement implies that $L = GK$ will also have a RHP-pole at $p$. Therefore, using equation (2) we have that,

$$
D(p) = 0 \Rightarrow T(p) = 1, \quad S(p) = 0
$$

Exercise 2  Stable Plant with no RHP Zeros

The magnitudes of the open-loop transfer function, $L_1(j\omega)$, and the closed-loop sensitivity function,

$$
S_1(j\omega) = \frac{1}{1 + L(j\omega)},
$$

are plotted on a logarithmic scale in Figure 1.

The open-loop plant contains no RHP zeros and the closed-loop system is stable. According to Slide 4.10 the Bode sensitivity integral is

$$
\int_0^\infty \ln|S(j\omega)|dw = \begin{cases} 
\pi \sum_{i=1}^{N_p} \text{Re}(p_i) & \text{if } L(s) \text{ is unstable,} \\
0 & \text{if } L(s) \text{ is stable,}
\end{cases}
$$

(3)
Figure 1: Magnitude of $L_1(j\omega)$ (dashed line) and $S_1(j\omega)$ (solid line) versus frequency $\omega$ [rad/sec]

where $p_1, \ldots, p_{N_p}$ are the RHP poles of the unstable open-loop transfer function $L(s)$. As $L_1(s)$ has no RHP poles, the Bode sensitivity integral is zero for this plant,

$$
\int_0^\infty \ln |S_1(j\omega)| \, dw = 0. \tag{4}
$$

By plotting the magnitude of the sensitivity, $S_1(j\omega)$, on a linear frequency scale in Figure 2 we graphically verify this result. Furthermore, numeric integration gives a value of $6 \cdot 10^{-4}$ for (4).
Exercise 3 Stable Plant with RHP Zero

The magnitudes of the open-loop transfer function, $L_2(j\omega)$, and the closed-loop sensitivity function, $S_2(j\omega)$, are plotted on a logarithmic scale in Figure 3. The open-loop plant contains a RHP zero at $s = 4$ and the closed-loop system is stable. According to Slide 4.14 the Bode sensitivity integral for a system with a RHP zero is

$$\int_0^\infty \ln |S(j\omega)| w(z_0, \omega)\, dw = \pi \ln \prod_{i=1}^{N_p} \left| \frac{p_i + z_0}{p_i - z_0} \right|,$$

where $p_1, \ldots, p_{N_p}$ are the RHP poles and $z_0$ is the RHP zero of the open-loop transfer function $L(s)$. The weighting function, $w(z_0, \omega)$, is defined as

$$w(z_0, \omega) = \frac{2z_0}{z_0^2 + \omega^2}.$$

Since $L_2(s)$ has no RHP poles the sensitivity integral is zero for this particular plant,

$$\int_0^\infty \ln |S_2(j\omega)| w(4, \omega)\, dw = 0.$$

Figure 4 shows the weighted magnitude of the sensitivity, $S_2(j\omega)$, on a linear frequency scale. We obtain a numeric integration value of $5 \cdot 10^{-4}$ for (6).
Exercise 4 Unstable Plant with no RHP Zeros

The magnitudes of the open-loop transfer function, \( L_3(j\omega) \), and the closed-loop sensitivity function, \( S_3(j\omega) \), are plotted on a logarithmic scale in Figure 5. The open-loop plant contains no RHP zeros and the closed-loop system is stable. The open-loop plant \( L_3(s) \) does however have a RHP pole at \( s = 1 \). From (3) we conclude the Bode sensitivity integral for this plant is

\[
\int_0^\infty \ln |S_3(j\omega)| \, dw = \pi \sum_{i=1}^{N_p} \text{Re}(p_i) = \pi.
\]

(7)

Figure 6 shows the magnitude of the sensitivity, \( S_3(j\omega) \), on a linear frequency scale. We obtain a numeric integration value of 3.1370 for (7).
Exercise 5  Unstable Plant with RHP Zero

The magnitudes of the open-loop transfer function, $L_4(j\omega)$, and the closed-loop sensitivity function, $S_4(j\omega)$, are plotted on a logarithmic scale in Figure 7. The open-loop plant contains a RHP zero at $s = 4$ and a RHP pole at $s = 1$. The closed-loop system is stable. From (5) we know that the Bode sensitivity integral for this plant is

$$\int_0^\infty \ln |S_4(j\omega)| w(4, \omega) \, dw = \pi \ln \left| \frac{1 + 4}{1 - 4} \right| = 1.6048.$$  \hspace{1cm} (8)

Figure 8 shows the weighted magnitude of the sensitivity, $S_4(j\omega)$, on a linear frequency scale. We obtain a numeric integration value of 1.5993 for (8).
Figure 5: Magnitude of $L_3(j\omega)$ (dashed line) and $S_3(j\omega)$ (solid line) versus frequency $\omega$ [rad/sec]

Figure 6: $\ln|S_3(j\omega)|$ versus frequency $\omega$ [rad/sec]
Figure 7: Magnitude of $L_4(j\omega)$ (dashed line), $S_4(j\omega)$ (solid line), and $w(4, \omega)$ (dash-dot line) versus frequency $\omega$ [rad/sec]

Figure 8: $\ln |S_4(j\omega)|$ versus frequency $\omega$ [rad/sec]