Exercise 1  Singular Values, Stability, Unstructured Model Uncertainty

1. For the first part of the exercise we consider an unstructured perturbation $\Delta$.

   a) In order to obtain the conditions for robust stability, we transform the given system into the usual $M\Delta$-structure shown in Figure 1.

   \[
   \begin{align*}
   z &= -GK(z + v) \\
   z &= -(I + GK)^{-1}GKv = -To v.
   \end{align*}
   \]

   Figure 1: Modified block diagram for robust stability.

   To this end, we have to compute the closed loop transfer function from $z$ to $v$ as shown in Figure 2:

   \[
   z = -GK(z + v) \implies z = -(I + GK)^{-1}GKv = -To v. \quad (1)
   \]

   From equation (1) it follows that $M = -To$. We need to verify that $To$ is stable. We can do so using Matlab or computing the closed loop eigenvalues. Since $\Delta$ is unstructured we know that we obtain robust stability if and only if

   \[
   \sigma(To(j\omega)\Delta(j\omega)) < 1 \quad \forall \omega = \frac{1}{\alpha}, \quad (2)
   \]

   which can be equivalently written as

   \[
   \frac{1}{\sigma(\Delta(j\omega))} \quad \forall \omega \quad \sigma(To(j\omega)) < \sigma(\Delta(j\omega)). \quad (3)
   \]
b) The easiest way to calculate the complementary sensitivity $T(s)$ in this case is over the state space representation. For the plant $G(s)$ we have

$$
\begin{align*}
\dot{x} &= Ax + Bu, \\
y &= Cx + Du,
\end{align*}
$$

where $A, B, C, D$ are given in Exercise 1. Substituting $u = K(r - y)$ with $K = I$, and $y = Cx$ then yields:

$$
\begin{align*}
\dot{x} &= (A - BKC)x + BKr \\
y &= (C - DKC)x + DKr
\end{align*}
$$

For the given matrices $A, B, C, D$ we thus obtain

$$
\begin{bmatrix}
A - BC \\
(C - DKC)
\end{bmatrix}
\begin{bmatrix}
B \\
DK
\end{bmatrix} =
\begin{bmatrix}
0 - 1 & a - a & 1 & 0 \\
-a + a & 0 - 1 & 0 & 1 \\
1 & a & 0 & 0 \\
-a & 1 & 0 & 0
\end{bmatrix}
$$

(4)

$$
\begin{bmatrix}
0 - 1 & a - a & 1 & 0 \\
-a + a & 0 - 1 & 0 & 1 \\
1 & a & 0 & 0 \\
-a & 1 & 0 & 0
\end{bmatrix} =
\begin{bmatrix}
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 \\
1 & a & 0 & 0 \\
-a & 1 & 0 & 0
\end{bmatrix}
$$

(5)

This can now be transformed into the following transfer function,

$$
T_o(s) = C(sI - A + BC)^{-1}B = \frac{1}{s + 1} \begin{bmatrix}
1 & a \\
-a & 1
\end{bmatrix}.
$$

(6)

It is a straightforward exercise to plot the maximum singular value $\sigma(T_o(j\omega))$ over frequency. The plot is shown in Figure 3.

c) From equation (3) it is clear where the admissible singular values of the uncertainty $\Delta$ have to lie. Namely, they are upper bounded by the inverse of the maximum singular value of $T$, as shown in Figure 3.

2. For the second part of the exercise we consider a structured $\Delta$.

a) In order to find the new stability condition we still look at the $M\Delta$ formulation of figure 1. Since the uncertainty is structured we now have the following test

$$
\mu_\Delta(M(j\omega)) \leq \frac{1}{\sigma(\Delta(j\omega))} = \frac{1}{\alpha},
$$

(7)

where, as before, $M = -T_o$. 

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Figure 2: Output multiplicative uncertainty.
Figure 3: Maximum singular values of the complementary sensitivity $T(s)$.

b) The test in (7) is less stringent than the test in (3) as $\mu_\Delta(M(j\omega)) \leq \sigma(M(j\omega))$. This makes sense as in the structured case we have “less” uncertainty than in the full block case, we then expect robust stability to be achieved more easily for the structured case.
Exercise 2  Generalized Plant

1. The given transfer functions just have to be arranged such that

\[
\begin{bmatrix}
y - r \\
u \\
r \\
y_m
\end{bmatrix} = P \begin{bmatrix} d \\ r \\ n \\ u \end{bmatrix},
\]

which yields the following expression for the generalized plant

\[
P = \begin{bmatrix} G_d & -I & 0 & G \\ 0 & 0 & 0 & I \\ 0 & I & 0 & 0 \\ G_d & 0 & I & G \end{bmatrix}.
\]

2. Following the hint, we assume \( u = K_1r + K_2y_m \). (a) From the block diagram,

\[
\begin{bmatrix} y - r \\ u \end{bmatrix} = \left[ I - GK_2 \quad 0 \right]^{-1} \begin{bmatrix} G_d & G(K_1 + K_2) - I & GK_2 \\ K_2G_d & K_1 & K_2 \end{bmatrix} \begin{bmatrix} d \\ r \\ n \end{bmatrix},
\]

and thus

\[
N = \left[ I - GK_2 \quad 0 \right]^{-1} \begin{bmatrix} G_d & G(K_1 + K_2) - I & GK_2 \\ K_2G_d & K_1 & K_2 \end{bmatrix}.
\]

(b) Using \( N = F_1(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \), first we need partition \( P \) as

\[
P_{11} = \begin{bmatrix} G_d & -I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P_{12} = \begin{bmatrix} G \\ I \end{bmatrix},
\]

\[
P_{21} = \begin{bmatrix} 0 & I & 0 \\ G_d & 0 & I \end{bmatrix}, \quad P_{22} = \begin{bmatrix} 0 \\ G \end{bmatrix}.
\]

With this we obtain exactly the same result as in (a).

Exercise 3  Cascade Implementation, Generalized Plant

Using the standard notation, we have that

\[
w = \begin{bmatrix} d_1 \\ d_2 \\ r \end{bmatrix}, \quad z = y_1 - r, \quad v = \begin{bmatrix} r - y_1 \\ y_2 \end{bmatrix}, \quad u = u_2.
\]

Since \( u = K_2(K_1(r - y_1) - y_2) \), the generalized controller is given by \( K = [K_2K_1 - K_2] \). Moreover, with

\[
y_1 - r = d_1 + G_1(d_2 + G_2u_2) - r, \quad \text{and} \quad y_2 = d_2 + G_2u_2
\]
we obtain the generalized plant as
\[ P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \]
where
\[ P_{11} = \begin{bmatrix} I & G_1 & -I \\ & I & \end{bmatrix}, \quad P_{12} = G_1 G_2, \]
\[ P_{21} = \begin{bmatrix} -I & -G_1 & I \\ 0 & I & 0 \end{bmatrix}, \quad P_{22} = \begin{bmatrix} -G_1 G_2 \\ G_2 \end{bmatrix}. \]

**Exercise 4  Robust Stability/Performance (Exam 2010, Problem 6)**

a) A block diagram of the control loop with the generalized plant \( P \) indicated is depicted in Figure 4.

![Block diagram of the control loop.](image)

Figure 4: Block diagram of the control loop.

The signals of the general control configuration are
\[ w = r, \quad z = e, \quad v = e, \quad u = u, \quad u_\Delta = u_A, \quad y_\Delta = y_A. \]  
(8)

Hence the generalized controller can be stated directly as
\[ K = K_1. \]  
(9)

The generalized plant \( P \) is composed of the terms
\[
\begin{align*}
\frac{y_\Delta}{u_\Delta} &= 0, & \frac{y_\Delta}{w} &= 0, & \frac{y_\Delta}{u} &= W_A, \\
\frac{z}{u_\Delta} &= -I, & \frac{z}{w} &= I, & \frac{z}{u} &= -G_0, \\
\frac{v}{u_\Delta} &= -I, & \frac{v}{w} &= I, & \frac{v}{u} &= -G_0.
\end{align*}
\]
and can be stated as
\[
P = \begin{bmatrix}
0 & 0 & W_A \\
-I & I & -G_0 \\
-I & I & -G_0
\end{bmatrix}.
\]  
\tag{10}

(0.5 points for each term in K and P)

b) \(N\) can be computed by the lower fractional transformation:

\[
N = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}
= \begin{bmatrix}
0 & 0 \\
-I & I
\end{bmatrix}
+ \begin{bmatrix}
W_A \\
-G_0
\end{bmatrix}
K(I + G_0K)^{-1}
\begin{bmatrix}
-I & I
\end{bmatrix}
\]
\[
= \begin{bmatrix}
-W_AK(I + G_0K)^{-1} & W_AK(I + G_0K)^{-1} \\
-I + G_0K(I + G_0K)^{-1} & I - G_0K(I + G_0K)^{-1}
\end{bmatrix}
\]
\[
= \begin{bmatrix}
-W_AKS_0 & W_AKS_0 \\
-S_0 & S_0
\end{bmatrix}
\]

(0.5 points for the lower LFT and for each term of \(N\))

c) (i) \(NS \iff N\) is internally stable.

Since there are no RHP pole/zero cancellations between the blocks, and since \(W_A(s)\) is stable, internal stability of \(N\) can be determined by checking stability of the nominal sensitivity function
\[
S_0(s) = (I + G_0K_1)^{-1} = \frac{(s + 2)(s + 1)}{k(s + 2) + (s + 1)(s + 2)} = \frac{s + 1}{s + 1 + \frac{1}{k}}.
\]

The nominal sensitivity function is stable for \(k > 1\).

(0.5 points approach, 0.5 points result)

(ii) \(NP \iff \|N_{22}\|_\infty < 1\) and NS.

\[
\|N_{22}\|_\infty = \|S_0\|_\infty = \left\|\frac{s + 1}{s + 1 + \frac{1}{k}}\right\|_\infty < 1 \iff \left|\frac{j\omega + 1}{j\omega + 1 + \frac{k}{s}}\right|_\infty < 1 \forall \omega.
\]

Hence we require \(k > 0\) for nominal performance.

(0.5 points approach, 0.5 points result)

(iii) \(RS \iff \|N_{11}\|_\infty < 1\) and NS.

\[
\|N_{11}\|_\infty = \|W_AKS_0\|_\infty = \left\|\frac{s + 2}{s + 1 + \frac{1}{k}}\right\|_\infty = \left\|\frac{k}{s + 1 + \frac{1}{k}}\right\|_\infty < 1
\]

The maximum is attained at steady state.
\[ |k| < |1 + k| \iff \begin{cases} 
  k < 1 + k, & 0 \leq k, \\
  -k < 1 + k, & -1 \leq k \leq 0, \\
  -k < -1 - k, & k \leq -1, \\
  0 < 1, & 0 \leq k, \\
  -1/2 < k, & -1 \leq k \leq 0, \\
  0 < -1, & k \leq -1, 
\end{cases} \]

This condition is fulfilled for all \( k > -1/2 \).

(1.5 points approach, 0.5 points result)

(iv) For \( k = -0.5 \), the system does not achieve NP, i.e. it will also not achieve RP.

(0.5 points answer, 0.5 points justification)