Questions

Exercise 1 [25%]
Consider the subset \( V \) of the normed space \( (C([-\pi,\pi],\mathbb{R}), \| \cdot \|_{\infty}) \) consisting of all the functions of the form
\[
f(t) = a \cos(t) + b \sin(t)
\]
(a) [6%] Show that \( V \) is a linear subspace of \( (C([-\pi,\pi],\mathbb{R}), \| \cdot \|_{\infty}) \). Which values of \( a \) and \( b \) lead to the zero vector?
(b) [6%] Show that \( \{ \cos(\cdot), \sin(\cdot) \} \) is a basis of \( V \). Is \( V \) finite dimensional? If so, what is its dimension?
(c) [6%] Show that the function \( \mathcal{A} : (V, \| \cdot \|_{\infty}) \to (\mathbb{R}^2, \| \cdot \|_2) \) defined by
\[
\mathcal{A}[f] = \begin{bmatrix} f(0) \\ f(\pi/2) \end{bmatrix}
\]
is linear.
(d) [7%] Compute the induced norm of the function \( \mathcal{A} : (V, \| \cdot \|_{\infty}) \to (\mathbb{R}^2, \| \cdot \|_2) \) defined in Part (c).

Hint: Recall that the induced norm is defined by
\[
\|\mathcal{A}\| = \sup_{f \neq 0} \frac{\|\mathcal{A}[f]\|_2}{\|f\|_{\infty}}
\]
To compute \( \|f\|_{\infty} \) for \( f \in V \), one may consider that
\[
a \cos(t) + b \sin(t) = \sqrt{a^2 + b^2} \sin(t + \varphi),
\]
for a certain \( \varphi \) such that \( \sin \varphi = \frac{a}{\sqrt{a^2 + b^2}} \) and \( \cos \varphi = \frac{b}{\sqrt{a^2 + b^2}} \).

Exercise 2 [25%]

Consider two linear time invariant systems
\[
\Sigma_1 \begin{cases} 
\dot{x}_1(t) = A_1 x_1(t) + B_1 u_1(t) \\
y_1(t) = C_1 x_1(t)
\end{cases} \quad \text{and} \quad \Sigma_2 \begin{cases} 
\dot{x}_2(t) = A_2 x_2(t) + B_2 u_2(t) \\
y_2(t) = C_2 x_2(t) + D_2 u_2(t)
\end{cases}
\]
where \( x_1(t) \in \mathbb{R}^{n_1}, x_2(t) \in \mathbb{R}^{n_2}, u_1(t) \in \mathbb{R}^{m_1}, u_2(t) \in \mathbb{R}^{m_2}, y_1(t) \in \mathbb{R}^{p_1}, y_2(t) \in \mathbb{R}^{p_2} \), and all matrices have appropriate dimensions.
(a) [8%] Consider the case where \( p_1 = m_2 \) and \( p_2 = m_1 \). Assume that the two systems are interconnected as shown in Figure 1, by setting \( u_2(t) = y_1(t) \) and \( u_1(t) = r(t) - y_2(t) \), where \( r(t) \in \mathbb{R}^{m_1} \) is a reference input signal. Derive a state space model for the resulting system with input \( r(t) \) and output \( y_1(t) \). Is the resulting system linear? Is it time invariant? What is the dimension of its state space?
(b) [8%] Consider now the case \( n_1 = 2, n_2 = 1, \) and \( m_1 = m_2 = p_1 = p_2 = 1 \). Assume that system \( \Sigma_1 \) is defined by the matrices
\[
A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}
\]
Under what conditions on \( A_2, B_2, C_2, D_2 \in \mathbb{R} \) will the interconnection of Figure 1 be controllable (from \( r(t) \))? Under what conditions will it be observable (from \( y_1(t) \))? 
(c) [9%] In the same setting as Part (b), set \( B_2 = 1 \). Is it possible to select values \( A_2, C_2, D_2 \in \mathbb{R} \) so that the eigenvalues of the resulting system are all equal to \(-1\)? Is the same true if \( B_2 = 0 \)?
Exercise 3 [25%]
Consider the linear time invariant system
\[ \dot{x}(t) = Ax(t) + Bu(t), \]
where \( x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, A \in \mathbb{R}^{n \times n}, \) and \( B \in \mathbb{R}^{n \times m} . \)

(a) [5%] Under what condition on \((A, B)\) is it possible to construct an input trajectory \( u(\cdot) : [0, 1] \to \mathbb{R}^m \) that drives the system from an arbitrary initial state \( x(0) \) to \( x(1) = 0 \)? Give a precise mathematical condition involving \( A \) and \( B \).

(b) [6%] Suppose that the pair \((A, B)\) satisfies the condition in part (a), and suppose that \( x(0) \neq 0 \in \mathbb{R}^n \). Is it possible to construct a controller of the form \( u(t) = Kx(t) \) for some \( K \in \mathbb{R}^{m \times n} \) such that \( x(1) = 0 \)? Justify your answer.

(c) [7%] Under what conditions on \((A, B)\) is it possible to construct a controller \( u(t) = Kx(t) \) with \( K \in \mathbb{R}^{n \times m} \) such that the system (†) is asymptotically stable? Explain any difference from the condition in (a) above.

(d) [7%] Suppose that the pair \((A, B)\) satisfies the condition in part (a), and suppose that \( x(0) \) is known. Given an arbitrary \( \bar{x} \in \mathbb{R}^n \), is it always possible to construct an input trajectory \( u(\cdot) : [0, \infty) \to \mathbb{R}^m \) such that \( x(t) = \bar{x} \) for all \( t \geq 1 \)? If your answer is “yes”, justify. If your answer is “no”, describe the set of \( \bar{x} \) for which this is possible.

Exercise 4 [25%]
Suppose that the linear time invariant system
\[ \dot{x}(t) = Ax(t) + Bu(t), \]
\[ y(t) = Cx(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t) \]
represents the attitude dynamics of a hypothetical satellite designed by ACME\textsuperscript{TM} Corp., where you are employed in the Control Division.

(a) [9%] Design a gain matrix \( L \in \mathbb{R}^{3 \times 1} \) such that the error dynamics of the observer
\[ \dot{\hat{x}} = A\hat{x} + Bu(t) + L(y(t) - C\hat{x}(t)) \]
have all eigenvalues equal to \(-1\).

(b) [6%] Your colleague Federico, who is in charge of stabilizing the system, has designed a feedback controller of the form \( u(t) = Kx(t) \), and he claims that he was able to make all the eigenvalues of \( A + BK \) equal to \(-2\). Prompted by this, the team leader John asks you to re-design your observer so that the error dynamics also have all their eigenvalues equal to \(-2\). Is it possible that Federico is telling the truth? Can you do what John is asking for?

(c) [10%] Your colleague Debasish from the Electronics Division comes up with a new sensor that can be used to directly measure the third state of the system. The model of the resulting satellite now becomes
\[ \dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t) \]
\[ y(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t) \]
Is it now easier to satisfy John’s requirement from Part (b)? Design, if possible, a gain matrix \( L \in \mathbb{R}^{3 \times 2} \) such that the eigenvalues of the observation error dynamics are all equal to \(-2\).