Exercise 1 [25%]
Consider the subset $V$ of the normed space $(C([-\pi, \pi], \mathbb{R}), \| \cdot \|_\infty)$ consisting of all the functions of the form

$$f(t) = a \cos(t) + b \sin(t)$$

(a) [6%] Show that $V$ is a linear subspace of $(C([-\pi, \pi], \mathbb{R}), \| \cdot \|_\infty)$. Which values of $a$ and $b$ lead to the zero vector?

(b) [6%] Show that $\{ \cos(\cdot), \sin(\cdot) \}$ is a basis of $V$. Is $V$ finite dimensional? If so, what is its dimension?

(c) [6%] Show that the function $A : (V, \| \cdot \|_\infty) \rightarrow (\mathbb{R}^2, \| \cdot \|_2)$ defined by

$$A[f] = \begin{bmatrix} f(0) \\ f(\pi/2) \end{bmatrix}$$

is linear.

(d) [7%] Compute the induced norm of the function $A : (V, \| \cdot \|_\infty) \rightarrow (\mathbb{R}^2, \| \cdot \|_2)$ defined in Part (c).

Hint: Recall that the induced norm is defined by

$$\|A\| = \sup_{f \neq 0} \frac{\|A[f]\|_2}{\|f\|_\infty}$$

To compute $\|f\|_\infty$ for $f \in V$, one may consider that

$$a \cos(t) + b \sin(t) = \sqrt{a^2 + b^2} \sin(t + \varphi),$$

for a certain $\varphi$ such that $\sin \varphi = \frac{a}{\sqrt{a^2 + b^2}}$ and $\cos \varphi = \frac{b}{\sqrt{a^2 + b^2}}$.

Solution

(a) To show that $V$ is a linear subspace, we must show that it is itself a linear space if endowed with the operations of the surrounding space (pointwise addition and multiplication by a scalar). Another way to say this is that $V$ must be closed under linear combinations taken in the surrounding space. Indeed, if $f, g \in V$ and $\alpha, \beta \in \mathbb{R}$,

$$[\alpha f + \beta g](t) = \alpha(a_f \cos t + b_f \sin t) + \beta(a_g \cos t + b_g \sin t)$$

$$= (\alpha a_f + \beta a_g) \cos t + (\alpha b_f + \beta b_g) \sin t$$

$$= \tilde{a} \cos t + \tilde{b} \sin t$$

and of course $\tilde{a} \cos(\cdot) + \tilde{b} \sin(\cdot) \in V$.

Suppose now that $a \cos(\cdot) + b \sin(\cdot) = \theta$, that is $a \cos t + b \sin t = 0$ for all $t$. In particular this implies $a \cos 0 + b \sin 0 = 0$, or $a = 0$, and $a \cos \frac{\pi}{2} + b \sin \frac{\pi}{2} = 0$, or $b = 0$.

(b) That $\{ \cos(\cdot), \sin(\cdot) \}$ generate $V$ is given by the definition of $V$. That $\{ \cos(\cdot), \sin(\cdot) \}$ are linearly independent follows from the second part of point (1). Therefore $\{ \cos(\cdot), \sin(\cdot) \}$ is a basis of $V$, $V$ is finite-dimensional and its dimension is 2.
(c) For all \( f, g \in V \),
\[
A[\alpha f + \beta g] = \begin{bmatrix} (\alpha f + \beta g)(0) \\ (\alpha f + \beta g)(\pi/2) \end{bmatrix} \\
= \alpha \begin{bmatrix} f(0) \\ f(\pi/2) \end{bmatrix} + \beta \begin{bmatrix} g(0) \\ g(\pi/2) \end{bmatrix} \\
= \alpha A[f] + \beta A[g]
\]

Therefore, \( A \) is linear.

(d) Clearly, following the hint,
\[
\|a \cos(\cdot) + b \sin(\cdot)\|_\infty = \sqrt{a^2 + b^2} \sup_{t \in [-\pi, \pi]} |\sin(t + \varphi)| = \sqrt{a^2 + b^2}
\]

Moreover,
\[
A[a \cos(\cdot) + b \sin(\cdot)] = \begin{bmatrix} a \cos 0 + b \sin 0 \\ a \cos \pi/2 + b \sin \pi/2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2
\]

hence
\[
\|A[a \cos(\cdot) + b \sin(\cdot)]\|_2 = \sqrt{a^2 + b^2}
\]

Therefore
\[
\|A[f]\|_2 = \|f\|_\infty \ \forall f \in V
\]

It immediately follows that \( \|A\| = 1 \).

Exercise 2 [25%]

Consider two linear time invariant systems
\[
\Sigma_1 \begin{cases} 
\dot{x}_1(t) = A_1x_1(t) + B_1u_1(t) \\
y_1(t) = C_1x_1(t)
\end{cases} \quad \text{and} \quad \Sigma_2 \begin{cases} 
\dot{x}_2(t) = A_2x_2(t) + B_2u_2(t) \\
y_2(t) = C_2x_2(t) + D_2u_2(t)
\end{cases}
\]

where \( x_1(t) \in \mathbb{R}^{n_1}, x_2(t) \in \mathbb{R}^{n_2}, u_1(t) \in \mathbb{R}^{m_1}, u_2(t) \in \mathbb{R}^{m_2}, y_1(t) \in \mathbb{R}^{p_1}, y_2(t) \in \mathbb{R}^{p_2}, \) and all matrices have appropriate dimensions.

(a) [8%] Consider the case where \( p_1 = m_2 \) and \( p_2 = m_1 \). Assume that the two systems are interconnected as shown in Figure 1, by setting \( u_2(t) = y_1(t) \) and \( u_1(t) = r(t) - y_2(t) \), where \( r(t) \in \mathbb{R}^{m_1} \) is a reference input signal. Derive a state space model for the resulting system with input \( r(t) \) and output \( y_1(t) \). Is the resulting system linear? Is it time invariant? What is the dimension of its state space?
(b) [8%] Consider now the case $n_1 = 2$, $n_2 = 1$, and $m_1 = m_2 = p_1 = p_2 = 1$. Assume that system $\Sigma_1$ is defined by the matrices
\[
A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
C_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]
Under what conditions on $A_2, B_2, C_2, D_2 \in \mathbb{R}$ will the interconnection of Figure 1 be controllable (from $r(t)$)? Under what conditions will it be observable (from $y_1(t)$)?

(c) [9%] In the same setting as Part (b), set $B_2 = 1$. Is it possible to select values $A_2, C_2, D_2 \in \mathbb{R}$ so that the eigenvalues of the resulting system are all equal to $-1$? Is the same true if $B_2 = 0$?

Solution

(a) Stacking the equations of $\Sigma_1$ and $\Sigma_2$, and doing the proper substitutions, we obtain
\[
\begin{align*}
\dot{x}_1(t) &= A_1 x_1(t) + B_1 (r(t) - y_2(t)) \\
&= A_1 x_1(t) - B_1 (C_2 x_2(t) + D_2 y_1(t)) + B_1 r(t) \\
&= (A_1 - B_1 D_2 C_1) x_1(t) - B_1 C_2 x_2(t) + B_1 r(t) \\
\dot{x}_2(t) &= A_2 x_2(t) + B_2 y_1(t) \\
&= B_2 C_1 x_1(t) + A_2 x_2(t) \\
y_1(t) &= C_1 x_1(t)
\end{align*}
\]
that is:
\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} = \begin{bmatrix}
A_1 - B_1 D_2 C_1 & -B_1 C_2 \\
B_2 C_1 & A_2
\end{bmatrix} \begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} + \begin{bmatrix}
B_1 \\
0
\end{bmatrix} r(t)
\]
\[
y_1(t) = \begin{bmatrix}
C_1 & 0
\end{bmatrix} \begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix}
\]
The resulting system is linear and time-invariant. The dimension of its state space is $n_1 + n_2$.

(b) The model reduces to
\[
\begin{align*}
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} &= \begin{bmatrix}
0 & 1 \\
D_2 & 0
\end{bmatrix} \begin{bmatrix}
0 \\
-C_2
\end{bmatrix} \begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} r(t) \\
y_1(t) &= \begin{bmatrix}
1 & 0
\end{bmatrix} \begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix}
\end{align*}
\]
or
\[
\begin{align*}
\dot{\xi}(t) &= \begin{bmatrix}
0 & 1 & 0 \\
D_2 & 0 & -C_2 \\
B_2 & 0 & A_2
\end{bmatrix} \xi(t) + \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} r(t) \\
y_1(t) &= \begin{bmatrix}
1 & 0 & 0
\end{bmatrix} \xi(t)
\end{align*}
\]
By checking the controllability matrix:
\[
C = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & D_2 \\
0 & 0 & B_2
\end{bmatrix}
\]
the system is controllable iff $B_2 \neq 0$. By checking the observability matrix:

$$O = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ D_2 & 0 & -C_2 \end{bmatrix}$$

the system is controllable iff $C_2 \neq 0$.

(c) Setting $B_2 = 1$, the system matrix of the connected system becomes:

$$A_F = \begin{bmatrix} 0 & 1 & 0 \\ D_2 & 0 & -C_2 \\ 1 & 0 & A_2 \end{bmatrix}$$

Its characteristic polynomial is

$$\chi(s) = \det \begin{bmatrix} s & -1 & 0 \\ -D_2 & s & C_2 \\ -1 & 0 & s - A_2 \end{bmatrix} = s^2(s-A_2)+C_2-D_2(s-A_2) = s^3-A_2s^2-D_2s+(C_2+D_2A_2)$$

It is easily seen that, assigning proper values to $A_2, C_2, D_2$ it is possible to assign the polynomial arbitrarily; in particular to assign $\chi(s) = (s+1)^3 = s^3 + 3s^2 + 3s + 1$ one has to choose $A_2 = -3$, $D_2 = -3$, and $C_2 = -8$.

On the other hand, if $B_2 = 0$,

$$\chi(s) = \det \begin{bmatrix} s & -1 & 0 \\ -D_2 & s & C_2 \\ 0 & 0 & s - A_2 \end{bmatrix} = s^2(s-A_2) - D_2(s-A_2) = (s^2-D_2)(s-A_2)$$

and it is not possible to assign the characteristic polynomial arbitrarily (two of them are either real and opposite, or complex conjugate).

Exercise 3 [25%]

Consider the linear time invariant system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (\dagger)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$, and $B \in \mathbb{R}^{n \times m}$.

(a) [5%] Under what condition on $(A, B)$ is it possible to construct an input trajectory $u(\cdot) : [0, 1] \to \mathbb{R}^m$ that drives the system from an arbitrary initial state $x(0)$ to $x(1) = 0$? Give a precise mathematical condition involving $A$ and $B$.

(b) [6%] Suppose that the pair $(A, B)$ satisfies the condition in part (a), and suppose that $x(0) \neq 0 \in \mathbb{R}^n$. Is it possible to construct a controller of the form $u(t) = Kx(t)$ for some $K \in \mathbb{R}^{m \times n}$ such that $x(1) = 0$? Justify your answer.

(c) [7%] Under what conditions on $(A, B)$ is it possible to construct a controller $u(t) = Kx(t)$ with $K \in \mathbb{R}^{n \times m}$ such that the system $(\dagger)$ is asymptotically stable? Explain any difference from the condition in (a) above.

(d) [7%] Suppose that the pair $(A, B)$ satisfies the condition in part (a), and suppose that $x(0)$ is known. Given an arbitrary $\bar{x} \in \mathbb{R}^n$, is it always possible to construct an input trajectory $u(\cdot) : [0, \infty) \to \mathbb{R}^m$ such that $x(t) = \bar{x}$ for all $t \geq 1$? If your answer is “yes”, justify. If your answer is “no”, describe the set of $\bar{x}$ for which this is possible.
Solution (a): The condition is that $(A, B)$ must be controllable. A mathematical condition is $\text{rank}[B, AB, \ldots, A^{n-1}B] = n$.

(b): No, since the closed-loop system matrix $A + BK$ is at best exponentially stable, and therefore, trajectories cannot reach 0 in finite time.

(c): The pair $(A, B)$ must be stabilizable, i.e., the unstable eigenvalues of $A$ must be controllable through $B$.

(d): “Staying still” at a given $\bar{x}$ means the vector-field must vanish at $\bar{x}$, i.e., $A\bar{x} + Bu(t) = 0$ for all $t > 1$. But this possible provided $A\bar{x}$ is in the column space of $B$. Thus, unless $B$ has rank $n$, the answer is in general “no”.

Exercise 4 [25%]
Suppose that the linear time invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$y(t) =Cx(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x(t)$$

represents the attitude dynamics of a hypothetical satellite designed by ACME™ Corp., where you are employed in the Control Division.

(a) [9%] Design a gain matrix $L \in \mathbb{R}^{3 \times 1}$ such that the error dynamics of the observer

$$\dot{\hat{x}} = A\hat{x} + Bu(t) + L(y(t) - C\hat{x}(t))$$

have all eigenvalues equal to $-1$.

(b) [6%] Your colleague Federico, who is in charge of stabilizing the system, has designed a feedback controller of the form $u(t) = Kx(t)$, and he claims that he was able to make all the eigenvalues of $A + BK$ equal to $-2$. Prompted by this, the team leader John asks you to re-design your observer so that the error dynamics also have all their eigenvalues equal to $-2$. Is it possible that Federico is telling the truth? Can you do what John is asking for?

(c) [10%] Your colleague Debasish from the Electronics Division comes up with a new sensor that can be used to directly measure the third state of the system. The model of the resulting satellite now becomes

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t)$$

Is it now easier to satisfy John’s requirement from Part (b)? Design, if possible, a gain matrix $L \in \mathbb{R}^{3 \times 2}$ such that the eigenvalues of the observation error dynamics are all equal to $-2$.

Solution
(1) Let
\[ L = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \]
We must design \( L \) such that \( A - LC \) has its eigenvalues at \(-1\). Now
\[ LC = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & l_2 & 0 \\ 0 & l_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & l_1 & 0 \\ 0 & l_2 & 0 \\ 0 & l_3 & 0 \end{bmatrix} \]
\[ A - LC = \begin{bmatrix} 0 & 1 - l_1 & 0 \\ 1 & -l_2 & 0 \\ 2 & -l_3 & -1 \end{bmatrix} \]
It is easily seen that \( l_3 \) has no effect on the eigenvalues; on the other hand, it suffices now to choose \( l_1 \) and \( l_2 \) such that the upper-left 2\( \times \)2 block has eigenvalues at \(-1\). This is easily done by noticing that the characteristic polynomial of such block is \( s^2 + l_2 s + (l_1 - 1) \), and should be equal to \((s + 1)^2 = s^2 + 2s + 1\). Hence, we choose \( l_2 = 2, l_1 = 2 \).

(2) Federico is correct, because the system is controllable. Indeed, its controllability matrix
\[ C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & -2 \end{bmatrix} \]
has clearly full rank.

On the other hand John is wrong, because the system is not observable. Indeed, its observability matrix
\[ O = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \]
has clearly rank 2.

(3) Now we can do what John was asking the new system being observable, since its observability matrix
\[ O = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ \cdots & \cdots & \cdots \end{bmatrix} \]
has full rank. Let
\[ L = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \\ l_5 \\ l_6 \end{bmatrix} \]
We must design \( L \) such that \( A - LC \) has its eigenvalues at \(-2\). Now
\[ LC = \begin{bmatrix} l_1 & l_4 \\ l_2 & l_5 \\ l_3 & l_6 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & l_1 & l_4 \\ 0 & l_2 & l_5 \\ 0 & l_3 & l_6 \end{bmatrix} \]
\[ A - LC = \begin{bmatrix} 0 & 1 - l_1 & -l_4 \\ 1 & -l_2 & -l_5 \\ 2 & -l_3 & -1 - l_6 \end{bmatrix} \]
We can see that $l_3, l_4, l_5$ can be safely chosen equal to zero, the other coefficients being sufficient to allocate the eigenvalues (we use $l_1, l_2$ to allocate the first two like we did before, and now we can use $l_6$ to choose the third). The characteristic polynomial of the upper-left $2 \times 2$ block is is $s^2 + l_2 s + (l_1 - 1)$, and should be equal to $(s + 2)^2 = s^2 + 4s + 4$. Hence, we choose $l_2 = 4, l_1 = 5$. Finally, the lower-right entry should also be equal to $-2$, hence we choose $l_6 = 1$. 