Exercise 1 [25%]  
Consider the following transfer function:

\[ G(s) = \frac{1}{s^2 + 4s + 3} \]

(a) [7%] Show that

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} -4 & 1 & -3 \\ 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)
\end{align*}
\]

is a state space realization of \( G(s) \). What is the dimension of this realization? Is the realization in controllable canonical form?

(b) [6%] Design an output feedback controller of the form

\[ u(t) = -ky(t) + r(t) \]

(where \( r(t) \) is an auxiliary input) and select the gain \( k \) so that the poles of the closed loop system are both at \( s = -2 \).

(c) [5%] Assume that you would like to improve your controller so that the poles of the closed loop system are all at \( s = -3 \). Can you achieve this by tuning the output feedback controller of Part (b)? If not, how would you modify your design? (You do not need to provide a complete design.)

(d) [7%] Using first principles modeling, your friend from EPFL derived the following dynamics for the system in question:

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} -4 & 1 & -3 \\ 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x(t)
\end{align*}
\]

Is this also a realization of the transfer function \( G(s) \)? Why is it different from the realization in Part (a)? If your friend is right, what would this imply about the performance of your controller in Parts (b) and (c)?

Hint: It is not necessary to completely invert a matrix in order to compute a transfer function. Recall that if \( A = [A_{ij}] \) is a \( 3 \times 3 \) matrix, \((A^{-1})_{31} = A_{21}A_{32} - A_{22}A_{31}\).

Exercise 2 [25%]  
Let \((H, \mathbb{C}, \langle \cdot, \cdot \rangle)\) be a Hilbert space, and \( \mathcal{A} : H \to H \) be a linear map. Recall that an eigenvalue of \( \mathcal{A} \) is a scalar \( \lambda \in \mathbb{C} \) such that \( \mathcal{A}(v) = \lambda v \) for some vector \( v \in V \), \( v \neq 0 \), which is called an eigenvector with respect to \( \lambda \).

(a) [5%] Suppose that \( \mathcal{A} \) is invertible and \( \mathcal{A}^{-1} \) is its inverse (you can take for granted that \( \mathcal{A}^{-1} \) is also linear). Show that, if \( \lambda \) is an eigenvalue of \( \mathcal{A} \), then \( \lambda \neq 0 \) and \( \lambda^{-1} \) is an eigenvalue of \( \mathcal{A}^{-1} \).

(b) [5%] Define \( p(\mathcal{A})(v) = p_0 \mathcal{I}(v) + p_1 \mathcal{A}(v) + p_2 \mathcal{A}^2(v) + \cdots + p_n \mathcal{A}^n(v) \), where \( p_0, p_1, \cdots, p_n \in \mathbb{C} \), \( \mathcal{I} \) is the identity map in \( H \), and \( \mathcal{A}^n(v) = \mathcal{A} \circ \mathcal{A} \circ \cdots \mathcal{A}(v) \), \( n \) times. Show that, if \( \lambda \) is an eigenvalue of \( \mathcal{A} \), then \( p(\lambda) = p_0 + p_1 \lambda + \cdots + p_n \lambda^n \) is an eigenvalue of \( p(\mathcal{A}) \). You may assume without proof that \( p(\mathcal{A}) : H \to H \) is linear.

(c) [5%] Suppose that \( v \) is an eigenvector of \( \mathcal{A} \) with respect to the eigenvalue \( \lambda \), and \( w \) is an eigenvector with respect to the eigenvalue \( \mu \). Show that, if \( \lambda \neq \mu \), then \( v \) and \( w \) are linearly independent.

(d) [5%] Suppose that \( \mathcal{A} \) is self-adjoint. Show that the eigenvalues of \( \mathcal{A} \) are real numbers.

(e) [5%] Suppose that \( \mathcal{A} \) is self-adjoint, \( v \) is an eigenvector of \( \mathcal{A} \) with eigenvalue \( \lambda \), and \( w \) is an eigenvector with eigenvalue \( \mu \). Show that, if \( \lambda \neq \mu \), then \( v \) and \( w \) are orthogonal.
Exercise 3 [25%]
Consider the following non-linear time varying system:

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = t \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \sin(x_1(t)) \\ \sin(x_2(t)) \end{bmatrix}$$ (1)

You are asked to analyze the stability properties at the origin. Having just completed the Linear System Theory course you decide to linearize the system and look at the stability properties of the linearization.

(a) [5%] Show that a first order linear approximation of system (1) around the origin is given by:

$$\dot{x}(t) = t \begin{bmatrix} -2 & -2 \\ 2 & -2 \end{bmatrix} x(t)$$

(b) [8%] Show that the state transition matrix of the linearized system is:

$$\Phi(t, 0) = e^{-t^2} \begin{bmatrix} \cos(t^2) & -\sin(t^2) \\ \sin(t^2) & \cos(t^2) \end{bmatrix}$$

(c) [5%] Using your answer in part (b) determine the stability of the origin for the linearized system.

(d) [7%] Consider now the time invariant systems:

$$\dot{x}(t) = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} x(t)$$

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x(t)$$

$$\dot{x}(t) = \begin{bmatrix} -2 & -2 \\ 2 & -2 \end{bmatrix} x(t)$$

obtained by freezing the time varying dynamics at instances $t = -1$, $t = 0$, and $t = 1$. Determine the stability properties of these three systems.

Exercise 4 [25%]
Suppose that the linear time invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) = \begin{bmatrix} -2 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = Cx(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x(t)$$

represents the attitude dynamics of a hypothetical satellite designed by Abbey Road Space Systems™, where you are employed in the Control Division.

(a) [6%] Is the satellite controllable? Is it stabilizable? Justify your answer in each case.

(b) [7%] The team manager John asks you to design a time-invariant state feedback controller of the form $u(t) = Kx(t)$ so that the closed loop system converges to the origin at the rate $e^{-2t}$? Is this a reasonable requirement? If so, design a feedback matrix $K \in \mathbb{R}^{1 \times 3}$ to implement it.

(c) [5%] John asked your colleague Paul to design an observer

$$\dot{x} = A\dot{x} + Bu(t) + L(y(t) - C\dot{x}(t))$$

so that the observation error decays to zero at the rate $e^{-3t}$. As Paul was unable to perform that task John decided to fire him. Help Paul save his job by explaining to John that his request was unreasonable.

(d) [7%] Your colleague Ringo from the Electronics Division comes up with a star tracker that can be used to directly measure the first state of the system. The model of the resulting satellite becomes

$$\dot{x}(t) = \begin{bmatrix} -2 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} x(t)$$

Is it now possible to satisfy the John’s requirement from Part (c)? Design, if possible, a gain matrix $L \in \mathbb{R}^{3 \times 2}$ so that the observation error decays to zero at the rate $e^{-3t}$.  