Signal- und Systemtheorie II
D-ITET, Semester 4

Notes 0: Introduction

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Where, when & what?

- **Lectures**
  - Thursdays, 8:00-10:00, ETF E1 (and today)

- **Examples classes:**
  - Mondays, 13:00-15:00, ETF C1 and ETF E1

- **Assessment**
  - Written examination
  - Optional mid-terms
  - March 23 and May 4
  - Mid term grade 15%+15%, only together, only positive
  - Usual exemptions (military, illness, with certificate) apply
  - Possibly unconventional format
Where, when & what?

• Exercises
  – Examples papers (discussed in examples classes)
  – Exercises in lecture notes
  – Neither graded since 2014
  – Integral part of the learning experience nonetheless
  – Example paper exercises in style of final exam questions
  – Please try to do them and discuss with instructor and assistants if you have questions
  – Please attend examples classes
  – Feel free to submit your solutions for grading
Reading material

• Lecture notes
  – Slides handout, available on class webpage & Moodle
  – Blackboard notes

• Recommended book
  – http://www.cds.caltech.edu/~murray/amwiki/index.php/Main_Page

• Other excellent books
  – G.F. Franklin, J.D. Powell, and A. Emami-Naeini, “Feedback control of dynamical systems”, Prentice-Hall, 2006 (also used in Regesysteme I/II)
The TORQUE concept

• ETH TORQUE pilot course in 2014
  – Tiny, Open-with-Restrictions courses focused on QUality and Effectiveness
  – “Flipped classroom” concept
  – Use of web and mobile technology before, during, and after lecture

• Available sources of online content, different purposes
  – ETH learning management tool (Moodle)
    • Our weekly off-line preparation tool for the class
  – ETH mobile application (EduApp)
    • Real time Q&A in the classroom
  – Experimental adaptive learning platform (Albie)
    • Exam preparation and interactive learning (trust Albie)
  – Class webpage
    • Just a repository with links to all of the above
Moodle: Learning management

• Official website for the Signals and Systems II TORQUE
  – https://moodle-app2.let.ethz.ch

• Log in using your ETH account and register for the Signals and Systems II TORQUE

• What you will find:
  – Short video tutorials on course material
  – Quizzes designed to test your understanding of course material
  – Forums to interact and ask questions about the course material (anonymous)

• How it will be used:
  – Videos and quizzes will be assigned before the lectures
  – The lecture will build on top of these assignments by adding more in depth material in a (hopefully) flipped classroom atmosphere
EduApp: Interactive lectures

- EduApp can be found at
  - [http://www.eduapp.ethz.ch](http://www.eduapp.ethz.ch)

- Install the app on your iPhone or Android mobile phone
- Log in using your ETH account – you should automatically see the SSII course if you are registered.

- What you will find:
  - An interactive platform that can be used during the lecture

- How it will be used:
  - Questions will be posed during some lectures and example sessions and students will be asked to contribute answers
  - Back channel available where students can ask questions anonymously
Albie (Optional, self paced study)

- The experimental platform Albie can be found at
  - http://www.albie.co  (Yes, that is .co NOT .com)

- Register using an ETH Zurich email address (must end in ethz.ch). After logging in for the first time, go to search and join the Signals and Systems course

- What you will find:
  - An experimental adaptive learning platform

- How it will be used:
  - Optional, not used in assignments during the semester
  - Personalized, non-linear content sequence
  - Last year many students used during their exam preparation
  - Search for content or “trust Albie” to tell you what to look at next
  - More: Learning statistics and comments attached to content
It’s all for a good cause!

• Class format new and experimental
• Much preparation, different concepts
• Please try to make the most of it
  – Watch the videos, do the quizzes, come prepared
  – Actively participate in the class, work on exercises, answer questions
  – Attend the examples classes where exam style questions will be answered
  – Provide feedback: What works, what does not
• If it all gets too much, play the SygSys game!
  – http://www.sigsystext.com/
Class content: Dynamical systems

- Describe evolution of variables over time
  - Input variables
  - Output variables
  - State variables

- Control:
  - Steer systems using inputs
  - Feedback
From signals to systems

SS1: System maps input signals to output signals

SS2: Where does input-output map come from?

RS1: What happens when we connect system inputs and outputs?
Dynamical systems

- Describe evolution of variables over time
  - Input variables
  - Output variables
  - State variables
- What is a “state”? 
- What values can it take?
- What is “time”?
- What values can it take?
- What is “evolution”?
- How can evolution be described?
Discrete vs continuous

• Discrete $\rightarrow$ Finite (or countable) values
  - $\{0, 1, 2, 3, \ldots\}$
  - $\{a, b, c, d\}$
  - $\{\text{ON, OFF}\}, \{\text{hot, warm, cool, cold}\}, \ldots$

• Continuous $\rightarrow$ Real values
  - $x \in \mathbb{R}, x \in \mathbb{R}^n$
  - $x \in [-1,1] \Rightarrow x \in \mathbb{R}, -1 \leq x \leq 1$
  - $\{(x_1,x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 \leq 1\}$

• Hybrid $\rightarrow$ Part discrete and part continuous
  - Airplane + flight management system
  - Thermostat + room temperature
## System classification (examples)

<table>
<thead>
<tr>
<th></th>
<th>Time State</th>
<th>Discrete</th>
<th>Continuous</th>
<th>Hybrid</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Discrete</strong></td>
<td></td>
<td>Finite state machines, Turing machines</td>
<td>Queuing systems</td>
<td></td>
</tr>
<tr>
<td><strong>Continuous</strong></td>
<td></td>
<td>( x_{k+1} = Ax_k + Bu_k ) ( y_k =Cx_k + Du_k )</td>
<td>Laplace transform ( \dot{x}(t) = Ax(t) + Bu(t) ) ( y(t) = Cx(t) + Du(t) )</td>
<td><strong>Impulse differential inclusions</strong></td>
</tr>
<tr>
<td><strong>Hybrid</strong></td>
<td></td>
<td>Mixed Logic-Dynamical systems</td>
<td>Switching diffusions</td>
<td>Hybrid automata</td>
</tr>
</tbody>
</table>
In this course

- We will concentrate mostly on
  - Continuous state
  - Continuous time
  - Linear systems

- We will also establish a connection to
  - Continuous state
  - Discrete time
  - Linear systems

  and to
  - Continuous state
  - Continuous time
  - Nonlinear systems

- Start with examples from many classes of systems
Course outline: Introductory material

1. Modeling
   - Mechanical and electrical systems
   - Discrete and continuous time systems
   - Discrete and continuous state systems
   - Linear and nonlinear (continuous state) systems

2. Revision: ODE and linear algebra
   - ODE = Ordinary Differential equations
   - Existence and uniqueness of solutions
   - Range and null spaces of matrices
   - Eigenvalues, eigenvectors, …
Course outline: Continuous time LTI

3. Time domain
   – LTI = Linear Time Invariant
   – State space equations
   – Time domain solution of state space equations

4. Controllability, observability, energy

5. “Frequency domain”
   – Revision of Laplace transforms
   – Laplace solution of state space equations
   – Stability
   – Bode and Nyquist plots
Course outline: Discrete time LTI and advanced topics

6. Discrete time LTI systems
   - Sampled data systems
   - Linear difference equations
   - Controllability and observability
   - \( z \)-transform
   - Simulation, Euler method and its stability

7. Nonlinear systems
   - Differences from linear systems
   - Multiple equilibria, limit cycles, chaos
   - Linearization
   - Stability
   - Examples
Notation

- \( \mathbb{Z} \) denotes the integers. This is a discrete set
  \[ \mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, \ldots \} \]
- \( \mathbb{N} \) denotes the natural numbers \( \mathbb{N} = \{0, 1, 2, \ldots \} \)
- \( \mathbb{C} \) denotes the complex numbers

\[
s = s_1 + js_2 \in \mathbb{C}
\]

\[
\text{Im}[s] = s_2
\]

\[
\text{Re}[s] = s_1
\]

\[
|s| = \sqrt{s_1^2 + s_2^2}
\]

\[
\angle s = \tan^{-1} \frac{s_2}{s_1}
\]

\[
s = |s| e^{j \angle s}
\]

\[
e^{j\theta} = \cos(\theta) + j\sin(\theta)
\]

Belongs to
Notation

- $\mathbb{R}^n$ denotes Euclidean space of dimension $n$. It is a finite dimensional vector space (sometimes called linear space). Special cases:
  - $n=1$, real line, $x \in \mathbb{R}$ (drop the superscript)
  - $n=2$, real plane,
    \[ x = (x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \]
  - General $n$, write $x$ as ordered list of numbers, or vector
    \[ x = (x_1, x_2, \ldots, x_n) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \]
Notation

• \( \mathbb{R}^{n \times m} \) matrices with \( n \) rows and \( m \) columns, whose elements are real

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1m} \\
a_{21} & a_{22} & \cdots & a_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nm}
\end{bmatrix} = \begin{bmatrix} a_{ij} \end{bmatrix}_{n \times m} \in \mathbb{R}^{n \times m}
\]

• Also a vector space, can define “length”, …

• Special cases \( \mathbb{R}^n = \mathbb{R}^{n \times 1}, \mathbb{R} = \mathbb{R}^{1 \times 1} \)

• Assume familiar with basic matrix operations (addition, multiplication, eigenvalues)
Notation

- Definition of sets \( \{ x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1 \} \)
- Special case: Intervals for \( a, b \in \mathbb{R}, a < b \)

\[
[a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \} \quad (a, b) = \{ x \in \mathbb{R} \mid a < x < b \}
\]

\[
[a, b) = \{ x \in \mathbb{R} \mid a \leq x < b \} \quad (a, b] = \{ x \in \mathbb{R} \mid a < x \leq b \}
\]

\[
(a, \infty) = \{ x \in \mathbb{R} \mid a < x \} \quad (-\infty, b] = \{ x \in \mathbb{R} \mid x \leq b \}
\]

\[ \mathbb{R}_+ = [0, \infty) \]

- \( \forall \) means “for all”
- \( \exists \) means “there exists”

**Exercise:** What do the sets \( \{ x \in \mathbb{R}^2 \mid x_1 = 0 \text{ or } x_2 = 0 \} \), \( \{ x \in \mathbb{R}^2 \mid |x_1| \geq |x_2| \} \) and \( \{ y \in \mathbb{R} \mid \exists x \in \mathbb{R}, y = x^2 \} \) look like?
# Notation

- **Continuous time** → \( t \in \mathbb{R}_+ \)
- **Discrete time** → \( k \in \mathbb{N} \)
- **Continuous state** → \( x \in \mathbb{R}^n \)
- **Continuous input** → \( u \in \mathbb{R}^m \)
- **Continuous output** → \( y \in \mathbb{R}^p \)

- **Discrete state** → \( q \in Q \)
  - e.g. thermostat → \( q \in Q = \{ON, OFF\} \)

- **Lower case letters:** Vectors/numbers \( x, u, t, k \)
- **Upper case letters:** Matrices, \( A, B, C, \ldots \)
Notation

• $f(\cdot): X \to Y$ function returning for $x \in X$ an $f(x) \in Y$
  \[ x \mapsto f(x) \]

• Example: Discrete time input signal
  \[ u(\cdot): \mathbb{N} \to \mathbb{R}^m \quad k \mapsto u(k) = u_k \]

  Discrete time
  Input at time $k$
  Shorthand notation

• Example: Continuous time state signal
  \[ x(\cdot): \mathbb{R}_+ \to \mathbb{R}^n \quad t \mapsto x(t) \]

  Continuous time
  State at time $t$
Linear functions: Euclidean space

- Special case: Linear function \( f(\bullet): \mathbb{R}^n \to \mathbb{R}^m \)
- For any \( x_1, x_2 \in \mathbb{R}^n, a_1, a_2 \in \mathbb{R} \)

\[
\begin{align*}
  f(a_1 x_1 + a_2 x_2) &= a_1 f(x_1) + a_2 f(x_2) \\
  \end{align*}
\]

- Multiplication by a matrix \( A \in \mathbb{R}^{m \times n} \), \( f(x) = Ax \)

\[
\begin{align*}
  f(a_1 x_1 + a_2 x_2) &= A(a_1 x_1 + a_2 x_2) = a_1 Ax_1 + a_2 Ax_2 = a_1 f(x_1) + a_2 f(x_2) \\
  \end{align*}
\]

- All linear functions on finite dimensional vector spaces can be written in this way

**Exercise:** Show that if \( f(\bullet): \mathbb{R}^n \to \mathbb{R}^m \), \( g(\bullet): \mathbb{R}^m \to \mathbb{R}^p \) are linear functions, then their composition \( g(f(\bullet)): \mathbb{R}^n \to \mathbb{R}^p \) is also linear. If \( f \) and \( g \) are defined in terms of matrices \( A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{p \times m} \) what does this composition correspond to?
Linear functions: Function spaces

• Linear functions defined more generally for vector (linear) spaces

• For example, for \( u_1(\cdot), u_2(\cdot): \mathbb{R} \rightarrow \mathbb{R}, a_1, a_2 \in \mathbb{R} \)
  define \( (a_1 u_1 + a_2 u_2)(\cdot): \mathbb{R} \rightarrow \mathbb{R} \) by
  \[
  (a_1 u_1 + a_2 u_2)(t) = a_1 u_1(t) + a_2 u_2(t) \quad \forall t \in \mathbb{R}
  \]

• Likewise, for \( U_1(\cdot), U_2(\cdot): \mathbb{C} \rightarrow \mathbb{C}, a_1, a_2 \in \mathbb{C} \)
  define \( (a_1 U_1 + a_2 U_2)(\cdot): \mathbb{C} \rightarrow \mathbb{C} \) by
  \[
  (a_1 U_1 + a_2 U_2)(s) = a_1 U_1(s) + a_2 U_2(s) \quad \forall s \in \mathbb{C}
  \]
Laplace transform and convolution

- Given $u(\cdot): \mathbb{R} \rightarrow \mathbb{R}$
- Laplace transform $U(\cdot): \mathbb{C} \rightarrow \mathbb{C}$
  \[ U(s) = \int_{-\infty}^{\infty} u(t)e^{-st} \, dt \]
- Convolution of $u$ with fixed function $h(\cdot): \mathbb{R} \rightarrow \mathbb{R}$
  \[ (u * h)(\cdot): \mathbb{R} \rightarrow \mathbb{R} \quad (u * h)(t) = \int_{-\infty}^{\infty} u(\tau)h(t - \tau) \, d\tau \]

**Exercise:** Show that the Laplace transform and the convolution are linear functions of $u(.)$
**Subtle points**

- In SSII interested in system response for positive times $t \in \mathbb{R}_+$
- Implicitly assume all signals = 0 for $t < 0$
- Hence Laplace transform simplifies to

$$U(s) = \int_{-\infty}^{\infty} u(t)e^{-st} \, dt = \int_{0}^{\infty} u(t)e^{-st} \, dt$$

(since $u(t)=0$ for $t < 0$)

- And convolution simplifies to

$$(u * h)(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau) \, d\tau = \int_{0}^{t} u(\tau)h(t-\tau) \, d\tau$$

(since $u(\tau) = 0$ for $\tau < 0$ and $h(t-\tau) = 0$ for $\tau > t$)
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Notes 1: Modeling

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Series of examples

1. **Pendulum**: Continuous time, continuous state, nonlinear autonomous system

2. **RLC circuit**: Continuous time, continuous state linear system with inputs

3. **Amplifier circuit**: Continuous time, continuous state linear system with inputs and outputs

4. **Population dynamics**: Discrete time, continuous state nonlinear system

5. **Manufacturing machine**: Discrete time, discrete state system

6. **Thermostat**: Continuous time, hybrid state system
Example 1: Pendulum

- A continuous time, continuous state, autonomous, nonlinear system
- Mass $m$ hanging from weightless string of length $l$
- String makes angle $\theta$ with downward vertical
- Friction with dissipation constant $d$
- Determine how the pendulum is going to move
- i.e. assuming we know where the pendulum is at “time” $t=0$ ($\theta_0$) and how fast it is moving ($\dot{\theta}_0$) determine where it will be at time $t$ ($\theta(t)$)
Pendulum: Equations of motion

• Evolution of $q$ governed by Newton’s law

\[ ml\ddot{\theta}(t) = -dl\dot{\theta}(t) - mg \sin \theta(t) \]

Exercise: Derive the differential equation from Newton’s laws of motion

• Need to solve Newton’s differential equation
• i.e. find a function $\theta(\bullet): \mathbb{R}_+ \rightarrow \mathbb{R}$

such that

\[ \theta(0) = \theta_0 \quad \dot{\theta}(0) = \dot{\theta}_0 \]

\[ \forall t \in \mathbb{R}_+, \quad ml\ddot{\theta}(t) = -dl\dot{\theta}(t) - mg \sin[\theta(t)] \]
Pendulum: Existence and uniqueness

• Such a function is known as a “solution” or a “trajectory” of the system

1. Does a trajectory exist for all $\theta_0, \dot{\theta}_0$?
2. Is there a unique trajectory for each $\theta_0, \dot{\theta}_0$?
3. Can we find this trajectory?

• Clearly important questions for differential equations used to model physical systems

• In general answer to questions may be “no”

• In fact, answer to question 3 usually is “no”!

• However, we can usually approximate the trajectory by computer simulation
Pendulum: MATLAB simulation

\[ l = 1, m = 1, d = 1, g = 9.8, \theta_0 = 0.75, \dot{\theta}_0 = 0 \]

function [xprime] = pendulum(t,x)
xprime=[0; 0];
l = 1;
m=1;
d=1;
g=9.8;
xprime(1) = x(2);
xprime(2) = -sin(x(1))*g/l-x(2)*d/m;

>> x=[0.75 0];
>> [T,X]=ode45('pendulum', [0 10], x');
>> plot(T,X);
>> grid;
Pendulum: State space description

- Convenient to write ODE more compactly

\[ \dot{x}(t) = f(x(t)), \quad x(t) \in \mathbb{R}^n \]

- For the pendulum, let

\[
x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \mathbb{R}^2 \quad \text{with} \quad x_1(t) = \theta(t), \ x_2(t) = \dot{\theta}(t)
\]

**Exercise**: A different \(f(x(t))\) would be obtained if \(x_1(t)\) and \(x_2(t)\) are selected differently.

Derive \(f(x(t))\) for

\[
x_1(t) = \theta^3(t) + \dot{\theta}(t), \ x_2(t) = \dot{\theta}(t)
\]

- Then

\[
\dot{x}(t) = \begin{bmatrix} x_2(t) \\ -\frac{d}{m} x_2(t) - \frac{g}{l} \sin x_1(t) \end{bmatrix} = f(x(t))
\]
Pendulum: State space description

- This first order ODE for \( x(t) \in \mathbb{R}^2 \) describes exactly the same “dynamics” as second order ODE for \( \theta(t) \in \mathbb{R} \)
- Vector \( x(t) \in \mathbb{R}^2 \) called the state of the system
- Function \( f(\bullet): \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) called the vector field
- For the pendulum \( f \) is a nonlinear function of \( x \)
- Solving Newton’s equation is equivalent to finding a function \( x(\bullet): \mathbb{R}_+ \rightarrow \mathbb{R}^2 \)

such that

\[
x(0) = \begin{bmatrix} \theta_0 \\ \dot{\theta}_0 \end{bmatrix} \\
\forall t \in \mathbb{R}_+, \quad \dot{x}(t) = f(x(t))
\]
Pendulum: Vector field & phase plane

Vector field $f(x)$

Trajectory $x_2(t)$ vs. $x_1(t)$

$\begin{align*}
\dot{x}_1 &= f(x_1, x_2) \\
\dot{x}_2 &= g(x_1, x_2)
\end{align*}$

$t = 0$
Example 2: RLC circuit

- Continuous time, continuous state, linear system
- Input voltage $v_1(t)$ (not autonomous)
- Determine evolution of voltages and currents

\[ v_R(t) - v_L(t) + v_{1}(t) - v_C(t) \]

\[ R \quad L \quad C \]

\[ i_L(t) \]
RLC circuit: Equations of “motion”

- From Kirchhoff’s laws + element equations
- E.g.
  \[ C \frac{dv_C(t)}{dt} = i_L(t) \]
  \[ L \frac{di_L(t)}{dt} = v_L(t) \]
  \[ v_R(t) = Ri_L(t) \]
  \[ v_L(t) = v_1(t) - v_R(t) - v_C(t) \]

  \[ \frac{d^2v_c(t)}{dt^2} + \frac{R}{L} \frac{dv_c(t)}{dt} + \frac{1}{LC} v_c(t) = \frac{1}{LC} v_1(t) \]

- Solution to ODE gives \( v_C(t) \)
- All other voltages and currents can be computed from \( v_C(t) \)
RLC circuit: MATLAB simulation

\[ R = 10 \quad L = 1 \]
\[ C = 0.01 \quad x_0 = 0 \]

Low pass filter

**Exercise:** Modify the MATLAB code for the pendulum to simulate the RLC circuit and generate such plots.
RLC circuit: State space description

- Try to write first order vector ODE $\dot{x}(t) = f(x(t)), x(t) \in \mathbb{R}^n$
- Based on our experience with the pendulum
  - Second order ODE for $v_C(t)$ suggests $x(t) \in \mathbb{R}^2$
  - $x(t)$ has something to do with energy
    - Potential ($\theta$) and kinetic ($\dot{\theta}$) in the pendulum
    - Stored in capacitor ($v_C$) and inductor ($i_L$) in circuit
- Try $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \mathbb{R}^2, \quad x_1(t) = v_C(t), \quad x_2(t) = i_L(t)$

(in circuits this usually works: Voltages across capacitors and currents through inductors can be selected as the states)

- Input voltage: External source of energy $u(t) = v_1(t)$
RLC circuit: State space description

- Relate state \( x(t) \) and input \( u(t) \)

\[
C \frac{dv_C(t)}{dt} = i_L(t) \Rightarrow \dot{x}_1(t) = \frac{1}{C} x_2(t)
\]

\[
L \frac{di_L(t)}{dt} = v_1(t) - v_R(t) - v_C(t) \Rightarrow \dot{x}_2(t) = \frac{1}{L} u(t) - \frac{R}{L} x_2(t) - \frac{1}{L} x_1(t)
\]

- In matrix form

\[
\dot{x}(t) = \begin{bmatrix}
0 & \frac{1}{C} \\
-\frac{1}{L} & -\frac{R}{L}
\end{bmatrix} x(t) + \begin{bmatrix}
0 \\
\frac{1}{L}
\end{bmatrix} u(t)
\]
RLC circuit: State space description

- Have written ODE of the form
  \[ \dot{x}(t) = Ax(t) + Bu(t) = f(x(t), u(t)) \]

- Similarities to pendulum
  - 2nd order ODE → two states
  - States related to energy stored in system

- Differences from pendulum
  - External source of energy → input \( u(t) \)
    → system no longer “autonomous”
  - Function \( f(x,u) \) linear in \( x \) and \( u \)
    → dynamics described by linear ODE

Exercise: What are the matrices \( A \) and \( B \)?
Example 3: Amplifier circuit

- Continuous time, continuous state linear system
- Input voltage $v_1(t)$
- Output voltage $v_0(t)$
Reminder: Operational amplifier (OpAmp)

External voltage source (not shown) provides energy for gain
Reminder: Ideal OpAmp

• Assume
  
  \[ R_{in} \approx \infty \quad \Rightarrow \quad i_{in} \approx 0 \]
  
  \[ R_{out} \approx 0 \quad \Rightarrow \quad i_{out} \text{ independent of } v_{out} \]
  
  \[ \mu \approx \infty \quad \Rightarrow \quad v_{in} \approx 0 \text{ if } v_{out} \text{ finite} \]

• “Virtual earth assumption”
• Makes circuit analysis much easier
• Note that
  – Input power \( i_{in}v_{in}=0 \)
  – Output power \( i_{out}v_{out} \) is arbitrary
• Necessary energy comes from external voltage source (not shown!)
Amplifier circuit: Equations of motion

- Assuming OpAmp is ideal

\[
\begin{align*}
C_1 \frac{dv_{c_1}(t)}{dt} &= i_1(t) \\
v_{in} \approx 0 &\Rightarrow i_1(t) = \frac{v_1(t) - v_{c_1}(t)}{R_1} \\
C_0 \frac{dv_{c_0}(t)}{dt} &= i_{c_0}(t) \\
i_{in} \approx 0 &\Rightarrow i_1(t) = i_{c_0}(t) + i_{R_0}(t) \\
i_{R_0}(t) &= \frac{1}{R_0} v_{c_0}(t)
\end{align*}
\]

\[
\begin{align*}
\frac{dv_{c_1}(t)}{dt} &= -\frac{v_{c_1}(t)}{R_1C_1} + \frac{v_1(t)}{R_1C_1} \\
\frac{dv_{c_0}(t)}{dt} &= -\frac{v_{c_0}(t)}{R_0C_0} - \frac{v_{c_1}(t)}{R_1C_0} + \frac{v_1(t)}{R_1C_0} \\
v_{in} \approx 0 &\Rightarrow v_0(t) = -v_{c_0}(t)
\end{align*}
\]
Amplifier circuit: State space description

• First order ODE in vector variables
• From our experience so far we would expect
  – Two state variables
  – Voltages of capacitors, $x_1(t)=v_{c1}(t)$, $x_2(t)=v_{c0}(t)$
  – One input variable, $u(t)=v_1(t)$
  – One output variable, $y(t)=v_0(t)$
• Write equations that relate input, states and output

\[
\frac{dx(t)}{dt} = \begin{bmatrix}
-\frac{1}{R_1C_1} & 0 \\
-\frac{1}{R_1C_0} & -\frac{1}{R_0C_0}
\end{bmatrix} x(t) + \begin{bmatrix}
\frac{1}{R_1C_1} \\
\frac{1}{R_1C_0}
\end{bmatrix} u(t)
\]

\[
y(t) = \begin{bmatrix}
0 & -1
\end{bmatrix} x(t)
\]
Amplifier circuit: State space description

- Have written in the form

\[ \dot{x}(t) = Ax(t) + Bu(t) = f(x(t),u(t)) \]
\[ y(t) = Cx(t) + Du(t) = h(x(t),u(t)) \]

- \( f \) and \( h \) are linear functions of the state and inputs

**Exercise**: What are the matrices \( A, B, C \) and \( D \)? What are their dimensions?

**Exercise**: Modify the matlab code given for the pendulum to simulate the amplifier circuit
Amplifier circuit: Simulation

Exercise: Why does the output settle to zero even though input is non-zero?
Amplifier circuit: Energy

- Energy stored in the system

\[ E(t) = \frac{1}{2} C_1 v_{C_1}^2(t) + \frac{1}{2} C_0 v_{C_0}^2(t) = \frac{1}{2} x(t)^T \begin{bmatrix} C_1 & 0 \\ 0 & C_0 \end{bmatrix} x(t) \]

- Quadratic function of the state

\[ E(t) = \frac{1}{2} x(t)^T Q x(t) \]

**Exercise:** What is the matrix \( Q \) in this case?

**Exercise:** Derive the energy equations for the pendulum and the RLC circuit. Can you find a matrix \( Q \) in both cases?
Amplifier circuit: Power

- Power: Instantaneous energy change

\[ P(t) = \frac{d}{dt} E(t) = \frac{1}{2} \dot{x}(t)^T Q x(t) + \frac{1}{2} x(t)^T Q \dot{x}(t) \]

\[ = \frac{1}{2} \left( x(t)^T A^T + u(t)^T B^T \right) Q x(t) + \frac{1}{2} x(t)^T Q \left( A x(t) + B u(t) \right) \]

\[ = \frac{1}{2} x(t)^T \left( A^T Q + QA \right) x(t) + \frac{1}{2} \left( u(t)^T B^T Q x(t) + x(t)^T Q B u(t) \right) \]

- Quadratic in state and input
- If there is no input \((u(t)=0)\)

\[ P(t) = \frac{1}{2} x(t)^T \left( A^T Q + QA \right) x(t) \]
Amplifier circuit: Power \((u(t)=0)\)

\[
P(t) = \frac{1}{2} x(t)^T \begin{bmatrix}
-\frac{1}{R_1 C_1} & -\frac{1}{R_1 C_0} \\
0 & -\frac{1}{R_0 C_0}
\end{bmatrix} \begin{bmatrix}
C_1 & 0 \\
0 & C_0
\end{bmatrix} + \begin{bmatrix}
C_1 & 0 \\
0 & C_0
\end{bmatrix} \begin{bmatrix}
-\frac{1}{R_1 C_1} & 0 \\
-\frac{1}{R_1 C_0} & -\frac{1}{R_0 C_0}
\end{bmatrix} x(t)
\]

\[
= \frac{1}{2} x(t)^T \begin{bmatrix}
-\frac{2}{R_1} & -\frac{1}{R_1} \\
\frac{1}{R_1} & -\frac{2}{R_0}
\end{bmatrix} x(t)
\]

\[
\Rightarrow P(t) = -\frac{x_1(t)^2}{R_1} - \frac{x_1(t)x_2(t)}{R_1} - \frac{x_2(t)^2}{R_0}
\]

Exercise: Derive this equation directly by differentiating the energy of the circuit

\[
E(t) = \frac{1}{2} C_1 v_{C_1}^2(t) + \frac{1}{2} C_0 v_{C_0}^2(t)
\]

Exercise: Repeat for the RLC circuit and pendulum
Population dynamics

• A discrete time, continuous state system
• Experiment:
  – Closed jar containing a number \( N \) of fruit flies
  – Constant food supply
• Question: How does fly population evolve?
• Number of flies limited by available food, epidemics
  – Few flies \( \rightarrow \) abundance of space/food \( \rightarrow \) more flies
  – Many flies \( \rightarrow \) competition for space/food \( \rightarrow \) fewer flies
• Maximum number “ecosystem” can support \( N_{\text{MAX}} \)
• State: Normalized population

\[
x = \frac{N}{N_{\text{MAX}}} \in [0,1]
\]
Population dynamics: State space model

- Track $x$ from generation to generation: $x_k$ population at generation $k$
- How does population at generation $k+1$ depend on $x_k$?
- Classic model: Logistic map

$$x_{k+1} = rx_k(1 - x_k) = f(x_k)$$

**Exercise:** Is the function $f(x)$ linear or non-linear?

**Exercise:** Show that if $r \in [0, 4]$ and $x_0 \in [0, 1]$ then $x_k \in [0, 1]$ for all $k=0, 1, 2, \ldots$
Population dynamics: Solution

- $r$ represents the “food” supply
  - Large $r$ means a lot of food is provided
  - Small $r$ means little food is provided
- Shape of $f(x)$ reflects population trends
  - Small population now $\rightarrow$ small population next generation (not enough individuals to breed)
  - Large population now $\rightarrow$ small population next generation (food/living space shortage, epidemics, etc.)
- How does the population change in time?
- This depends a lot on $r$
  1. If $r \in [0,1)$ then $x_k$ decays to 0 (i.e. all flies die)
  2. If $r \in [1,3)$ then $x_k$ tends to a steady state value (i.e. the fly population stabilizes)
  3. If $r \in [3,1+\sqrt{6})$ then $x_k$ tends to a 2-periodic solution (i.e. the population alternates between two values)
  4. Eventually chaotic behavior!
Population dynamics: Simulation

- $r = 0.5$
- $r = 1.5$
- $r = 3.3$
- $r = 3.99$
Manufacturing system

• A discrete time, discrete state system
• Model of a machine in a manufacturing shop
• Machine can be in three states
  – Idle (I)
  – Working (W)
  – Broken (D)
• State changes when certain “events” happen
  – A part arrives and starts getting processed (p)
  – The processing is completed and the part moves on to the next machine (c)
  – The machine fails (f)
  – The machine is repaired (r)
• Finite number of states and inputs:
  – Finite State machine or
  – Finite Automaton
Manufacturing system: State space model

- Possible states of the machine
  \[ q \in Q = \{ I, W, D \} \]
- Possible inputs (events)
  \[ \sigma \in \Sigma = \{ p, c, f, r \} \]
- Not all events are possible for all states, e.g.
  - A part cannot start getting processed (\( \sigma = p \)) while the machine is broken (\( q = D \))
  - The machine can only be repaired (\( \sigma = r \)) when broken (\( q = D \))
- Transition function \( \delta : Q \times \Sigma \to Q \)
- Write as discrete time system
  \[ q_{k+1} = \delta(q_k, \sigma_k) \]

Exercise: Is \( \delta \) linear or nonlinear? Does the question even make sense?
Manufacturing system: Automaton

\[ \delta(I, p) = W \]
\[ \delta(W, c) = I \]
\[ \delta(W, f') = D \]
\[ \delta(D, r) = I \]

- All other combinations not allowed
- Assume we start at \( I \)
- Easier to represent by a graph

**Exercise:** If graph has \( e \) arrows, how many lines are needed to define \( \delta \)?

**Exercise:** \( Q \) has \( n \) elements and \( \Sigma \) has \( m \) elements, how many lines are needed (at most) to define \( \delta \)?

\[ \text{Initial state} \]
Manufacturing system: Solution

• Assume initially $q_0 = I$. What are the sequences of events the machine can experience?

• Some sequences are possible while others are not
  – $pcp \rightarrow$ possible
  – $ppc \rightarrow$ impossible

• The set of all acceptable sequences is called the language of the automaton

• The following are OK
  – Arbitrary number of $pc$ denoted by $(pc)^*$
  – Arbitrary number of $pfr$ denoted by $(pfr)^*$
  – Possibly followed by a $p$ or $pf$

\[(p \cdot c + p \cdot f \cdot r)^* \cdot (1 + p + p \cdot f)\]
Thermostat

- A continuous time, hybrid system
- Thermostat is trying to keep the temperature of a room at around 20 degrees
  - Turn heater on if temperature $\leq 19$ degrees.
  - Turn heater off if temperature $\geq 21$ degrees.
- Due to uncertainty in the radiator dynamics, the temperature may fall further, to 18 degrees, or rise further, to 22 degrees
- Both a continuous and a discrete state
  - Discrete state: Heater $q(t) \in Q = \{ON, OFF\}$
  - Continuous state: Room temperature $x(t) \in \mathbb{R}$
- Evolution in continuous time
- No external input (autonomous system)
Thermostat: State space model

- Different differential equations for $x$, depending on ON or OFF
  - Heater OFF: Temperature decreases exponentially toward 0
    $$\dot{x}(t) = -\alpha x(t)$$
  - Heater ON: Temperature increases exponentially towards 30
    $$\dot{x}(t) = -\alpha (x(t) - 30)$$
- Heater changes from ON to OFF and back depending on $x(t)$
- Natural to describe by mixture of differential equation and graph notation

**Exercise:** Solve the differential equations to verify exponential increase/decrease.
Thermostat: Hybrid automaton

Initial state (OFF,20)

x(0) = 20

OFF

\dot{x}(t) = -ax(t)

x(t) \geq 18

Can stay OFF as long as ...

x(t) \leq 19

Can go ON provided that ...

ON

\dot{x}(t) = -a(x(t) - 30)

x(t) \leq 22

While OFF x(t) changes according to ...

x(t) \geq 21
Thermostat: Solutions

$q(t)$

$x(t)$
Continuous modeling: Generic steps

1. Identify input variables
   - Usually obvious
   - Quantities that come from the outside
   - Say \( m \) such input variables
   - Stack them in vector form, denote by \( u(t) \in \mathbb{R}^m \)

2. Identify output variables
   - Usually obvious
   - Quantities that can be measured
   - Say \( p \) such quantities
   - Stack them in vector form, denote by \( y(t) \in \mathbb{R}^p \)
Continuous modeling: Generic steps

3. Select state variables
   - Need to encapsulate the past
   - Need (together with inputs) to determine future
   - For physical systems often related to energy storage
   - For mechanical systems can usually select positions \((q(t))\) and velocities \((v(t))\)
   - For electrical circuits can usually select capacitor voltages \((v_C(t))\) and inductor currents \((i_L(t))\)
   - Other choices possible, may lead to simpler models
   - Say \(n\) such variables
   - Stack them in vector form, denote by \(x(t) \in \mathbb{R}^n\)
Continuous modeling: Generic steps

4. Take derivatives of states
   - Try to obtain $n$ equations with derivative of one state on the left hand side and a function of the states and inputs on the right hand side
   - For mechanical systems
     • Position derivatives easy, $\dot{q}(t) = v(t)$
     • Velocity derivatives (=accelerations) from Newton law
   - For electrical circuits
     • Current/voltage relations $C \frac{dv_C(t)}{dt} = i_C(t), \quad L \frac{di_L(t)}{dt} = v_L(t)$
     • Relate to each other by Kirchoff’s laws
   - Result: Vector differential equation $\dot{x}(t) = f(x(t), u(t))$
Continuous modeling: Generic steps

5. Write output variables as a function of state and input variables
   - Usually easy
   - Result: Vector algebraic equation $y(t) = h(x(t), u(t))$

6. Determine whether the system is linear, etc.
   - Are the functions $f$ and $h$ linear or not?

Disclaimer:
   - Generic steps seem easy, but require creativity!
   - Mathematical models never the same as reality
   - With any luck, close enough to be useful
Signal- und Systemtheorie II
D-ITET, Semester 4

Notes 2: Revision of ODE and linear algebra

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State space models

- For the time being continuous time, continuous state models
  - Nonlinear
  - Linear
- State equations are ordinary differential equations
  - Reminder of some tools to deal with these
- Continuous state models have extra structure
  - States, inputs, and outputs take values in vector spaces
- To exploit structure need tools from linear algebra
- Discrete time continuous state models very similar
  - State equations are difference equations, else the same
- Discrete state and hybrid models somewhat different
Assumed to be known

- Matrix product, compatible dimensions
  - Associative: \((AB)C = A(BC)\)
  - Distributive with respect to addition: \(A(B + C) = AB + AC\)
  - Non-commutative: \(AB \neq BA\) in general

- Transpose of a matrix
  - \((AB)^T = B^T A^T\)

- For square matrices
  - Identity matrix \(AI = IA = A\)
  - In every dimension there exists a unique identity matrix
  - Inverse matrix \(A^{-1}A = AA^{-1} = I\)
  - May not always exist
  - When it does it is unique
  - Computation of the determinant and its basic properties
The 2-norm

**Definition:** The 2-norm is a function \( \| \cdot \| : \mathbb{R}^n \rightarrow \mathbb{R} \) that to each \( x \in \mathbb{R}^n \) assigns a real number

\[
\| x \| = \sqrt{\sum_{i=1}^{n} x_i^2}
\]

**Fact 2.1:** For all \( x, y \in \mathbb{R}^n, a \in \mathbb{R} \)
1. \( \| x \| \geq 0 \) and \( \| x \| = 0 \) if & only if \( x = 0 \)
2. \( \| ax \| = |a| \cdot \| x \| \)
3. \( \| x + y \| \leq \| x \| + \| y \| \)

**Exercise:** Show that \( \| x \|^2 = x^T x \)

**Exercise:** Prove 1 and 2. Is the 2-norm a linear function?

- The 2-norm is a measure of “size” or “length”
- Distance between \( x, y \in \mathbb{R}^n \) is \( \| x - y \| \)
- The set of points that are closer than \( r > 0 \) to \( x \in \mathbb{R}^n \)

\[
\left\{ y \in \mathbb{R}^n \mid \| x - y \| < r \right\}
\]
**Inner product**

**Definition:** The **inner product** is a function \( \langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) that takes two vectors \( x, y \in \mathbb{R}^n \) and returns the real number
\[
\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i = x^T y
\]

**Fact 2.2:** For all \( x, y, z \in \mathbb{R}^n \), \( a, b \in \mathbb{R} \)
1. \( \langle x, y \rangle = \langle y, x \rangle \)
2. \( a \langle x, y \rangle + b \langle z, y \rangle = \langle ax + bz, y \rangle \)
3. \( \langle x, x \rangle = \|x\|^2 \)

**Exercise:** Prove these. For fixed \( y \), is the function \( \langle y, \cdot \rangle \) linear?

- **Orthogonal:** Meet at right angles
- **Orthonormal:** Pairwise orthogonal and unit length

**Exercise:** Are the vectors \(
\begin{bmatrix}
0 \\
2 \\
\end{bmatrix},
\begin{bmatrix}
1 \\
0 \\
\end{bmatrix}
\)
othogonal? Orthonormal?
Linear independence

**Definition:** A set of vectors \( \{x_1, x_2, \ldots, x_m\} \in \mathbb{R}^n \) is called **linearly independent** if for \( a_1, a_2, \ldots, a_m \in \mathbb{R} \)

\[
a_1 x_1 + a_2 x_2 + \cdots + a_m x_m = 0 \iff a_1 = a_2 = \cdots = a_m = 0
\]

Otherwise they are called linearly dependent.

• Linearly dependent if and only if at least one is a linear combination of the rest. E.g. if \( a_1 \neq 0 \)

\[
a_1 x_1 + a_2 x_2 + \cdots + a_m x_m = 0 \Rightarrow x_1 = -\frac{a_2}{a_1} x_2 - \cdots - \frac{a_m}{a_1} x_m
\]

**Fact 2.3:** There exists a set with \( n \) linearly independent vectors in \( \mathbb{R}^n \), but any set with more than \( n \) vectors is linearly dependent.

**Exercise:** What is a set of \( n \) linearly independent vectors of \( \mathbb{R}^n \)?
Subspaces

**Definition:** A set of vectors $S \subseteq \mathbb{R}^n$ is called a **subspace** of $\mathbb{R}^n$ if for all $x, y \in S$ and $a, b \in \mathbb{R}$, we have that $ax + by \in S$.

- Note that the set $S$ is generally an infinite set
- Examples of subspaces of $\mathbb{R}^n$
  - $S = \mathbb{R}^n$ and $S = \{0\}$
  - $\{x \in \mathbb{R}^n \mid x_1 = 2x_2\}$
  - $\{x \in \mathbb{R}^n \mid x_1 = 0\}$
- Not subspaces
  - $\{x \in \mathbb{R}^n \mid x_1 = 1\}$
  - $\{x \in \mathbb{R}^n \mid x_1 = 0 \text{ or } x_2 = 0\}$

**Exercise:** Draw these sets for $n=2$. Prove that they are/are not subspaces.
**Basis of a subspace**

**Definition:** The span of \( \{x_1, x_2, \ldots, x_m\} \subset \mathbb{R}^n \) is set of all linear combinations of these vectors.

- In fact, span = smallest subspace containing \( \{x_1, x_2, \ldots, x_m\} \)

**Exercise:** Show span is a subspace.

**Definition:** A set of vectors \( \{x_1, x_2, \ldots, x_m\} \subset \mathbb{R}^n \) is called a basis for a subspace \( S \subset \mathbb{R}^n \) if

1. \( \{x_1, x_2, \ldots, x_m\} \) are linearly independent
2. \( S = \text{span}\{x_1, x_2, \ldots, x_m\} \)

In this case, \( m \) is called the dimension of \( S \).

- All subspaces of \( \mathbb{R}^n \) have bases. The number of vectors, \( m \), in all bases of a given subspace is the same. By Fact 2.3, \( n \geq m \)
- Basis of subspace is not unique
- Different bases related through “change of coordinates”
Range space of a matrix

**Definition:** The **range space** of a matrix \( A \in \mathbb{R}^{n \times m} \) is the set
\[
\text{range}(A) = \{ y \in \mathbb{R}^n \mid \exists x \in \mathbb{R}^m, y = Ax \}.
\]

**Fact 2.4:** \( \text{range}(A) \) is a subspace of \( \mathbb{R}^n \)

**Definition:** The **rank** of a matrix \( A \in \mathbb{R}^{n \times m} \) is the dimension of \( \text{range}(A) \).

**Fact 2.5:** \( \text{range}(A) = \text{span}\{a_1, a_2, \ldots, a_m\} \).

\( \text{rank}(A) \) = number of linearly independent columns, \( a_1, \ldots, a_m \) of \( A \).

**Exercise:** Prove Fact 2.4

**Exercise:** Prove Fact 2.5
**Null space of a matrix**

**Definition:** The **null space** of a matrix \( A \in \mathbb{R}^{n \times m} \) is the set

\[
\text{null}(A) = \left\{ x \in \mathbb{R}^m \mid Ax = 0 \right\}
\]

**Fact 2.6:** \( \text{null}(A) \) is a subspace of \( \mathbb{R}^m \)

**Exercise:** Prove Fact 2.6

\[
A = \left[ \begin{array}{c}
\hat{a}_1^T \\
\vdots \\
\hat{a}_n^T 
\end{array} \right], \text{ where } \hat{a}_i^T = \left[ a_{i1} \ a_{i2} \cdots a_{im} \right]
\]

**Fact 2.7:** \( \text{null}(A) \) = set of vectors orthogonal to the rows of \( A \)

**Exercise:** Prove Fact 2.7

**Fact 2.8:** \( \text{rank}(A) \) = number of linearly independent rows, \( \hat{a}_1, \ldots, \hat{a}_n \) of \( A \)
Inverse of a square matrix

**Definition:** The inverse of a matrix $A \in \mathbb{R}^{n \times n}$ is a matrix $A^{-1} \in \mathbb{R}^{n \times n}$

$$A^{-1}A = AA^{-1} = I$$

**Fact 2.9:** If an inverse of $A$ exists then it is unique.

**Exercise:** Prove Fact 2.9

**Definition:** A matrix is called singular if it does not have an inverse. Otherwise it is called non-singular or invertible.

**Fact 2.10:** $A$ is invertible if and only if $\det(A) \neq 0$
Inverse of a square matrix

If $A$ is invertible

$$A^{-1} = \frac{\text{adj}(A)}{\det(A)}$$

Adjoint matrix = matrix of subdeterminants transposed

Exercise: Show that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \frac{1}{ad - bc}$$

Fact 2.11: $A$ is invertible if and only if the system of linear equations $Ax = y$ has a unique solution $x \in \mathbb{R}^n$ for all $y \in \mathbb{R}^n$

Fact 2.12: $A$ is invertible if and only if $\text{null}(A) = \{0\}$

Fact 2.13: $A$ is invertible if and only if $\text{range}(A) = \mathbb{R}^n$

Exercise: Prove Facts 2.11-2.12
Systems of linear equations

\[ Ax = y \quad A \in \mathbb{R}^{n \times m}, \ y \in \mathbb{R}^n \ \text{given} \]
\[ x \in \mathbb{R}^m \ \text{unknown} \]

- \( m=n \) unique solutions if and only if \( A \) invertible (Fact 2.11)
- If \( A \) singular system will have either no solutions, or infinite number of solutions
- \( n>m \rightarrow \) equations \( > \) unknowns \( \rightarrow \) generally no solution

**Fact 2.14**: If \( A \) has rank \( m \) then \( x = \left( A^T A \right)^{-1} A^T y \) is the unique minimizer of \( \|Ax - y\| \)

- \( n<m \rightarrow \) unknowns \( > \) equations \( \rightarrow \) generally infinite solutions

**Fact 2.15**: If \( A \) has rank \( n \) then the system has infinitely many solutions. The one with the minimum norm is \( x = A^T \left( AA^T \right)^{-1} y \)
Orthogonal matrices

**Definition:** A matrix is called **orthogonal** if $AA^T = A^T A = I$

**Fact 2.16:** $A$ is orthogonal if and only if its columns are ortho-normal. The product of orthogonal matrices is orthogonal. If $A$ is orthogonal then $\|Ax\| = \|x\|$

**Exercise:** Prove Fact 2.16
**Definition:** A (nonzero) vector \( w \in \mathbb{C}^n \) is called an *eigenvector* of a matrix \( A \in \mathbb{R}^{n \times n} \) if there exists a number \( \lambda \in \mathbb{C} \) such that
\[
Aw = \lambda w.
\]
The number \( \lambda \) is then called an *eigenvalue* of \( A \).

- Eigenvalues and eigenvectors are in general complex even if \( A \) is a real matrix.
- An \( n \times n \) matrix has \( n \) eigenvalues (some may be repeated).
  They are the solutions of the characteristic polynomial
  \[
  \det(\lambda I - A) = \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_n = 0
  \]
- The \( n \) eigenvalues of \( A \) are called the *spectrum* of \( A \).

**Exercise:** Show that if \( w \) is an eigenvector then so is \( aw \) for any \( a \in \mathbb{C} \).
Eigenvalues and eigenvectors

Theorem 2.1: (Cayley-Hamilton) Every matrix $A$ is a solution of its characteristic polynomial

$$A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_n I = 0$$

**Exercise:** Show that this implies that all powers $A^m$ for $m=0, 1, \ldots$ can be written as linear combinations of $I, A, A^2, \ldots, A^{n-1}$

Fact 2.17: $A$ is invertible if and only if all its eigenvalues are non-zero

**Exercise:** Show Fact 2.17 using Fact 2.12
Diagonalizable matrices

\[ Aw_i = \lambda_i w_i \Rightarrow A[ w_1 \ w_2 \ \ldots \ w_n ] = [ w_1 \ w_2 \ \ldots \ w_n ] \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} \]

\[ W \in \mathbb{C}^{n \times n} \]
\[ \Lambda \in \mathbb{C}^{n \times n} \]

Definition: A is called \textbf{diagonalizable} if \( W \) is invertible

Fact 2.18: If the eigenvalues of \( A \) are distinct (\( \lambda_i \neq \lambda_j \) if \( i \neq j \)) then its e-vectors are linearly independent

Exercise: Show that Fact 2.18 implies that \( W \) is invertible
Symmetric, positive definite and positive semi-definite matrices

**Definition:** A matrix is called **symmetric** if $A = A^T$.

**Fact 2.19:** Symmetric matrices have real eigenvalues and orthogonal eigenvectors.

**Exercise:** If $A$ is symmetric then there exist $U \in \mathbb{R}^{n \times n}$ orthogonal & $\Lambda \in \mathbb{R}^{n \times n}$ diagonal such that $A = U \Lambda U^T$.

**Definition:** A symmetric matrix is called **positive definite** if $x^T A x > 0$ for all $x \neq 0$. It is called **positive semi-definite** if $x^T A x \geq 0$.

**Fact 2.20:** A matrix is positive definite if and only if it has (real) positive e-values. It is positive semi-definite if and only if it has (real) non-negative e-values.

We use $A > 0$ (resp. $A \geq 0$) as a shorthand for $A$ symmetric, positive (semi-)definite.
Singular value decomposition

**Fact 2.21:** For any \( A \in \mathbb{R}^{n \times m} \)

\[
A = U \Sigma V^T
\]

where \( U \in \mathbb{R}^{n \times n} \) and \( V \in \mathbb{R}^{m \times m} \) are orthogonal and \( \Sigma \in \mathbb{R}^{n \times m} \) “diagonal” with non-negative elements.

The elements of \( \Sigma \) are called the **singular values** of \( A \).
State Space: Inputs, outputs and states

- Mathematical model of physical system described by
  - Input variables (denoted by $u_1, u_2, \ldots, u_m \in \mathbb{R}$)
  - Output variables (denoted by $y_1, y_2, \ldots, y_p \in \mathbb{R}$)
  - State variables (denoted by $x_1, x_2, \ldots, x_n \in \mathbb{R}$)
- All inputs states and outputs are real numbers
- Usually write more compactly as vectors

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \in \mathbb{R}^m \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} \in \mathbb{R}^p \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

Input vector \hspace{1cm} Output vector \hspace{1cm} State vector

**Exercise:** Which of the examples in Notes 1 can be described by real vectors? What are these vectors?

- Number of states, $n$, is called **dimension** (or **order**) of the system
State space: Dynamics

- Dynamics of process imply relations between variables
  - Differential equations giving evolution of states as a function of the states, inputs and possibly time, i.e. we have functions

\[ f_i(\cdot) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}, \quad \frac{d}{dt} x_i(t) = f_i(x(t),u(t),t), \quad i = 1,\ldots,n \]

  - Algebraic equations giving the outputs as a function of the states, inputs and possibly time, i.e. we have functions

\[ h_i(\cdot) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}, \quad y_i(t) = h_i(x(t),u(t),t), \quad i = 1,\ldots,p \]

- Equations usually come from “laws of nature”
  - Newton’s laws for mechanical systems
  - Electrical laws for circuits
  - Energy and mass balance for chemical reactions
In vector form

- Usually write more compactly be defining

\[ f(\cdot) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \rightarrow \mathbb{R}^n \quad h(\cdot) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \rightarrow \mathbb{R}^p \]

\[ f(x,u,t) = \begin{bmatrix} f_1(x,u,t) \\ \vdots \\ f_n(x,u,t) \end{bmatrix} \quad h(x,u,t) = \begin{bmatrix} h_1(x,u,t) \\ \vdots \\ h_p(x,u,t) \end{bmatrix} \]

- Then state, input and output vectors are linked by

\[ \frac{dx(t)}{dt} = f(x(t),u(t),t) \]
\[ y(t) = h(x(t),u(t),t) \]

- State space form
- \( f \) called the vector field

Exercise: What are the functions \( f \) for the pendulum, RLC and opamp examples of Notes 1? What are the dimensions of these systems?
**Linear and autonomous systems**

**Definition:** A system in state space form is called **autonomous** if it is time invariant and has no input variables

\[ \dot{x}(t) = f(x(t)), \quad y(t) = h(x(t)) \]

**Exercise:** Which of the systems considered in Notes 1 are autonomous? Which are time invariant?

**Definition:** A system in state space form is called **linear** if the functions \( f \) and \( h \) are linear, i.e.

\[ \dot{x}(t) = A(t)x(t) + B(t)u(t) \]

\[ y(t) = C(t)x(t) + D(t)u(t) \]

**Exercise:** Which of the systems considered in Notes 1 are linear?

**Exercise:** What are the general equations for a linear time invariant system?
Higher order differential equations

• Often “laws of nature” expressed in terms of higher order differential equations
  – For example, Newton’s law $\rightarrow$ second order ODE

• These can be converted to state space form by defining lower order derivatives (all except the highest) as states

Exercise: Convert the following differential equation of order $r$ into state space form

$$\frac{d^r y(t)}{dt^r} + g \left( y(t), \frac{dy(t)}{dt}, \ldots, \frac{d^{r-1} y(t)}{dt^{r-1}} \right) = 0$$

What is the dimension of the resulting system? Is it autonomous? Under what conditions is it linear?
Time invariant systems

- The explicit time dependence can be eliminated by introducing time as an additional state with
  \[ \dot{t} = 1 \]
- The result is a **time invariant system** of dimension \( n+1 \)

**Exercise:** Convert the following time varying system
\[
\begin{align*}
\dot{x}(t) &= f(x(t), u(t), t), \quad x \in \mathbb{R}^n \\
y(t) &= h(x(t), u(t), t)
\end{align*}
\]
of dimension \( n \) into a time invariant system of dimension \( n+1 \).

**Exercise:** Repeat for the linear time varying system
\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad x \in \mathbb{R}^n \\
y(t) &= C(t)x(t) + D(t)u(t)
\end{align*}
\]
Is the resulting time invariant system linear?
Coordinate transformation

- What happens if we perform a change of coordinates for the state vector?
- Restrict attention to time invariant linear systems
  \[ \dot{x}(t) = Ax(t) + Bu(t) \]
  \[ y(t) = Cx(t) + Du(t) \]
  and linear changes of coordinates
  \[ \hat{x}(t) = Tx(t), \quad T \in \mathbb{R}^{n \times n}, \det(T) \neq 0 \]
- In the new coordinates we get another linear system
  \[ \dot{\hat{x}}(t) = TAT^{-1}\hat{x}(t) + TBu(t) \]
  \[ y(t) = CT^{-1}\hat{x}(t) + Du(t) \]
- Could be useful, transformed system may be simpler
Solution of state space equations

• Only autonomous systems for the time being

\[ \dot{x}(t) = f(x(t)), \quad y(t) = h(x(t)) \]

• Non-autonomous systems essentially the same, formal mathematics more complicated

• What is the “solution” of the system?
  – Where do we start? Say \( x(t_0) = x_0 \in \mathbb{R}^n \), at time \( t_0 \in \mathbb{R} \)
  – How long do we go? Say until some \( t_1 \geq t_0 \)

• Would like to find functions

\[ x(\cdot) : [t_0, t_1] \rightarrow \mathbb{R}^n, \quad y(\cdot) : [t_0, t_1] \rightarrow \mathbb{R}^p \]

“satisfying” system equations for given conditions
Solution of state space equations

Definition: A pair of functions $x(\cdot): [t_0, t_1] \rightarrow \mathbb{R}^n, y(\cdot): [t_0, t_1] \rightarrow \mathbb{R}^p$ is a solution of the state space system over the interval $[t_0, t_1]$ starting at $x_0 \in \mathbb{R}^n$ if

1. $x(t_0) = x_0$
2. $\dot{x}(t) = f(x(t)), \ \forall t \in [t_0, t_1]$
3. $y(t) = h(x(t)), \ \forall t \in [t_0, t_1]$

• Note that $x(\cdot)$ implicitly defines $y(\cdot)$
• Therefore the difficulty is finding the solution to the differential equation
• Because the system is autonomous the starting time is also unimportant

Exercise: Show that $x(t)$ is a solution over the interval $[0, T]$ if and only if $x(t-t_0)$ is a solution over the interval $[t_0, t_0+T]$ starting at the same initial state.
Existence and uniqueness of solutions

- For autonomous systems the “only” thing we need to do is, given \( f(\cdot): \mathbb{R}^n \rightarrow \mathbb{R}^n, x_0 \in \mathbb{R}^n, T \geq 0 \), find a function \( x(\cdot): [0,T] \rightarrow \mathbb{R}^n \) such that

\[
x(0) = x_0 \quad \text{and} \quad \dot{x}(t) = f(x(t)), \quad \forall t \in [0,T]
\]

- Does such a function exist for some \( T \)?
- Is it unique, or can there be more than one?
- Do such functions exist for all \( T \)?
- Can we compute them even if they do?
- Clearly all these are important for physical models
- Unfortunately answer is sometimes “no”
Examples

• Illustrate potential problems on 1 dimensional systems
• **No solutions:** Consider the system

\[
\dot{x}(t) = -\text{sgn}(x(t)) = \begin{cases} 
-1 & \text{if } x(t) \geq 0 \\
1 & \text{if } x(t) < 0 
\end{cases}
\]

starting at \(x_0=0\). The system has no solution for any \(T\)

• **Many solutions:** Consider the system \(\dot{x}(t) = 3x(t)^{2/3}, x_0 = 0\). For any \(a \geq 0\) the following is a solution for the system

\[
x(t) = \begin{cases} 
(t-a)^3 & \text{if } t \geq a \\
0 & \text{if } t < a 
\end{cases}
\]

**Exercise:** Compute the solutions for \(x_0 = 1\) and \(x_0 = -1\). Are they defined for all \(T\)?

**Exercise:** Prove this is the case.
Examples

• **No solutions for \( T \) large:** Consider the system

\[
\dot{x}(t) = 1 + x(t)^2, x_0 = 0.
\]

• The solution is \( x(t) = \tan(t) \)

• So many things can go wrong!

• Fortunately many important systems are “well-behaved”

**Exercise:** Prove this. What happens at \( t=\pi/2 \)?

**Definition:** A function \( f : \mathbb{R}^n \to \mathbb{R}^n \) is called **Lipschitz** if there exists \( \lambda > 0 \) such that for all \( x, \hat{x} \in \mathbb{R}^n \)

\[
\|f(x) - f(\hat{x})\| \leq \lambda \|x - \hat{x}\|
\]
Lipschitz functions

- \( \lambda \) is called the Lipschitz constant of \( f \)
- Lipschitz functions are continuous but not necessarily differentiable
- All differentiable functions with bounded derivatives are Lipschitz
- Linear functions are Lipschitz

**Exercise:** Show that any other \( \lambda' \geq \lambda \) is also a Lipschitz constant

**Exercise:** Show that \( f(x) = |x| \) (\( x \) is a real number) is Lipschitz. What is the Lipschitz constant? Is the function differentiable?

**Exercise:** Show that \( f(x) = Ax \) is Lipschitz. What is a Lipschitz constant? Is the function differentiable?

**Exercise:** Show that the \( f(x) \) in the three pathological examples in p. 2.30-2.31 are not Lipschitz.
Existence and uniqueness

**Theorem 2.2:** If $f$ is Lipschitz then the differential equation

$$\dot{x}(t) = f(x(t)),$$

with initial condition $x_0 \in \mathbb{R}^n$

has a unique solution $x(\bullet): [0, T] \to \mathbb{R}^n$ for all $T \geq 0$

and all $x_0 \in \mathbb{R}^n$.

- So state space systems defined by Lipschitz vector fields are well behaved:
  - They have unique solutions
  - Over arbitrarily long horizons
  - Wherever they start
Continuity

• Even if a unique solution exists, this does not mean we can find it.

• Sometimes we can: See the pathological examples above and linear systems (Notes 3).

• Usually have to resort to simulation on computer.

• Construct approximate numerical solution.

• It helps if solutions that start close remain close.

**Theorem 2.3:** If $f$ is Lipschitz then the solutions starting at $x_0, \hat{x}_0 \in \mathbb{R}^n$ are such that for all $t \geq 0$

\[
\|x(t) - \hat{x}(t)\| \leq e^{\lambda t}\|x_0 - \hat{x}_0\|
\]

• Continuous dependence on initial condition.
Non-autonomous systems

• Formally, we would expect that given

\[ f(\cdot): \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \rightarrow \mathbb{R}^n, x_0 \in \mathbb{R}^n, t_1 \geq t_0 \geq 0, u(\cdot): [t_0, t_1] \rightarrow \mathbb{R}^m \]

the solution would be a function \( x(\cdot): [t_0, t_1] \rightarrow \mathbb{R}^n \)

\[ x(t_0) = x_0 \text{ and } \dot{x}(t) = f(x(t), u(t), t), \quad \forall t \in [t_0, t_1] \]

• This is OK if \( f \) is continuous in \( u \) and \( t \) and \( u(.) \) is continuous in \( t \)
• Unfortunately discontinuous \( u(.) \) are quite common
• Fortunately there is a fix, but the math is harder
• Roughly speaking need
  – \( f(x,u,t) \) Lipschitz in \( x \), continuous in \( u \) and \( t \)
  – \( u(t) \) continuous for “almost all” \( t \)

**Exercise:** What goes wrong in the case of discontinuity?
Signal- und Systemtheorie II
D-ITET, Semester 4

Notes 3: Continuous LTI systems in time domain

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LTI systems in state space form

- General state space systems

\[
\begin{align*}
\frac{d}{dt} x(t) &= f(x(t), u(t), t) \\
y(t) &= h(x(t), u(t), t)
\end{align*}
\]

\[
\begin{bmatrix}
u_1 \\ u_2 \\ \vdots \\ u_m
\end{bmatrix} \in \mathbb{R}^m \quad \begin{bmatrix}
y_1 \\ y_2 \\ \vdots \\ y_p
\end{bmatrix} \in \mathbb{R}^p \quad \begin{bmatrix}
x_1 \\ x_2 \\ \vdots \\ x_n
\end{bmatrix} \in \mathbb{R}^n
\]

- LTI systems are linear and time invariant, i.e.

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

\[
\begin{align*}
A &\in \mathbb{R}^{n \times n} \\
B &\in \mathbb{R}^{n \times m} \\
C &\in \mathbb{R}^{p \times n} \\
D &\in \mathbb{R}^{p \times m}
\end{align*}
\]
Block diagram representation
LTI systems in state space form

For LTI systems state space equations
- $n$ coupled, first order, linear differential equations
- $p$ linear algebraic equations
- Time invariant coefficients

\[
\begin{align*}
\dot{x}_1(t) &= a_{11}x_1(t) + \cdots + a_{1n}x_n(t) + b_{11}u_1(t) + \cdots + b_{1m}u_m(t) \\
\dot{x}_2(t) &= a_{21}x_1(t) + \cdots + a_{2n}x_n(t) + b_{21}u_1(t) + \cdots + b_{2m}u_m(t) \\
&\quad \vdots \\
\dot{x}_n(t) &= a_{n1}x_1(t) + \cdots + a_{nn}x_n(t) + b_{n1}u_1(t) + \cdots + b_{nm}u_m(t) \\
\end{align*}
\]

\[
\begin{align*}
y_1(t) &= c_{11}x_1(t) + \cdots + c_{1n}x_n(t) + d_{11}u_1(t) + \cdots + d_{1m}u_m(t) \\
y_2(t) &= c_{21}x_1(t) + \cdots + c_{2n}x_n(t) + d_{21}u_1(t) + \cdots + d_{2m}u_m(t) \\
&\quad \vdots \\
y_p(t) &= c_{p1}x_1(t) + \cdots + c_{pn}x_n(t) + d_{p1}u_1(t) + \cdots + d_{pm}u_m(t)
\end{align*}
\]

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
b_{11} & \cdots & b_{1m} \\
\vdots & \ddots & \vdots \\
b_{n1} & \cdots & b_{nm}
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
c_{11} & \cdots & c_{1n} \\
\vdots & \ddots & \vdots \\
c_{p1} & \cdots & c_{pn}
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
d_{11} & \cdots & d_{1m} \\
\vdots & \ddots & \vdots \\
d_{p1} & \cdots & d_{pm}
\end{bmatrix}
\]
# Examples

<table>
<thead>
<tr>
<th>RLC Circuit</th>
<th>Amplifier Circuit</th>
</tr>
</thead>
</table>
| \[
\begin{align*}
C \frac{dv_C(t)}{dt} &= i_C(t) = i_L(t) \\
L \frac{di_L(t)}{dt} &= v_L(t) = v_1(t) - v_R(t) - v_C(t)
\end{align*}
\] | \[
\begin{align*}
\frac{dv_{C_1}(t)}{dt} &= -\frac{v_{C_1}(t)}{RC_1} + \frac{v_1(t)}{RC_1} \\
\frac{dv_{C_0}(t)}{dt} &= -\frac{v_{C_0}(t)}{R_0C_0} - \frac{v_{C_1}(t)}{R_1C_0} + \frac{v_1(t)}{R_1C_0}
\end{align*}
\] |

\[
\dot{x}(t) = \begin{bmatrix}
0 & 1 \\
-\frac{1}{L} & -\frac{R}{L}
\end{bmatrix} x(t) + \begin{bmatrix}
0 \\
\frac{1}{L}
\end{bmatrix} u(t)
\]

\[
\dot{x}(t) = \begin{bmatrix}
-\frac{1}{R_1C_1} & 0 \\
-\frac{1}{R_1C_0} & -\frac{1}{R_0C_0}
\end{bmatrix} x(t) + \begin{bmatrix}
\frac{1}{R_1C_0} \\
\frac{1}{R_1C_0}
\end{bmatrix} u(t)
\]

\[
y(t) = \begin{bmatrix}
0 & -1
\end{bmatrix} x(t)
\]

cf. pendulum

\[
\dot{x}(t) = \begin{bmatrix}
x_2(t) \\
-\frac{d}{m} x_2(t) - \frac{g}{l} \sin x_1(t)
\end{bmatrix}
\]
System solution

• Since system is time invariant, assume we are given
  – Initial condition $x_0 \in \mathbb{R}^n$
  – The input values $u(\bullet) : [0, T] \rightarrow \mathbb{R}^m$

• Compute the system solution $x(\bullet) : [0, T] \rightarrow \mathbb{R}^n$

  $x(0) = x_0$ and $\dot{x}(t) = Ax(t) + Bu(t), \ \forall t \in [0, T]$

• If we can do this, then output $y(\bullet) : [0, T] \rightarrow \mathbb{R}^p$

  $y(t) = Cx(t) + Du(t), \ \forall t \in [0, T]$

• For simplicity assume input continuous function of time
State solution

The system solution is

\[ x(t) = \Phi(t)x_0 + \int_0^t \Phi(t - \tau)Bu(\tau) \, d\tau \]

where

\[ \Phi(t) = e^{At} = I + At + \frac{A^2t^2}{2!} + \ldots + \frac{A^k t^k}{k!} + \ldots \in \mathbb{R}^{n \times n} \]

and the integral is computed element by element

(cf. Taylor series expansion: \( e^{at} = 1 + at + \frac{a^2t^2}{2!} + \ldots \) if \( a \in \mathbb{R} \) )
Output solution

Simply combine state solution

\[ x(t) = \Phi(t)x_0 + \int_0^t \Phi(t - \tau)Bu(\tau) \, d\tau \]

with output map

\[ y(t) = Cx(t) + Du(t) \]

to obtain

\[ y(t) = C\Phi(t)x_0 + \int_0^t C\Phi(t - \tau)Bu(\tau) \, d\tau + Du(t) \]
Fact 3.1: The state transition matrix is such that

1. \( \Phi(0) = I \)

2. \( \frac{d}{dt} \Phi(t) = A\Phi(t) \)

3. \( \Phi(-t) = [\Phi(t)]^{-1} \)

4. \( \Phi(t_1 + t_2) = \Phi(t_1)\Phi(t_2) \)

Exercise: Prove properties 1, 2 and that (harder) \( \Phi(t)\Phi(-t) = \Phi(-t)\Phi(t) = I \)

Exercise: By invoking the existence discussion from Notes 2, show property 4 (harder).
Proof of solution formula (sketch)

• Clearly

\[ x(0) = \Phi(0)x_0 + \int_0^0 \Phi(0 - \tau)Bu(\tau)\,d\tau = x_0 \]

• To show that \( \dot{x}(t) = Ax(t) + Bu(t) \) use the Leibnitz formula for differentiating integrals

\[
\frac{d}{dt} \int_{g(t)}^{f(t)} l(t, \tau)\,d\tau = l(t, g(t)) \frac{d}{dt} g(t) \\
- l(t, f(t)) \frac{d}{dt} f(t) \\
+ \int_{g(t)}^{f(t)} \frac{\partial}{\partial t} l(t, \tau)\,d\tau
\]
Example: RC circuit

- Inputs: $u(t) = v_s(t)$
- States: $x(t) = v_C(t)$
- Initial condition: $x_0 = v_C(0)$
- State space equations

\[
\dot{x}(t) = -\frac{1}{RC}x(t) + \frac{1}{RC}u(t)
\]

- Response to step with amplitude $1V$

\[
x(t) = e^{-\frac{t}{RC}}x_0 + \left(1 - e^{-\frac{t}{RC}}\right)
\]

**Exercise:** Derive the state space equations. What are the “matrices” $A$ and $B$?

**Exercise:** Derive the step response.
State solution structure

The solution consists of two parts:

\[ x(t) = \Phi(t)x_0 + \int_0^t \Phi(t - \tau)Bu(\tau) d\tau \]

Total transition = Zero Input transition + Zero State transition

\[ u(t) = 0 \forall t \Rightarrow x(t) = ZIT \quad \text{Linear function of initial state} \]

\[ x_0 = 0 \Rightarrow x(t) = ZST \]

• Linear function of input
• Convolution integral
Superposition principle

- ZST linear in input trajectory
- ZST under input $u_1(\cdot):[0,t] \to \mathbb{R}^m$
  \[ x_1(t) = \int_0^t \Phi(t-\tau)Bu_1(\tau) d\tau \]
- ZST under input $u_2(\cdot):[0,t] \to \mathbb{R}^m$
  \[ x_2(t) = \int_0^t \Phi(t-\tau)Bu_2(\tau) d\tau \]
- ZST under input $u(\tau) = a_1u_1(\tau) + a_2u_2(\tau)$ for $\tau \in [0,t]$
  \[ a_1, a_2 \in \mathbb{R}, \quad x(t) = \int_0^t \Phi(t-\tau)Bu(\tau) d\tau \]
  \[ = \int_0^t \Phi(t-\tau)B(a_1u_1(\tau) + a_2u_2(\tau)) d\tau \]
  \[ = a_1x_1(t) + a_2x_2(t) \]
Output solution structure

The solution consists of two parts:

\[ y(t) = C\Phi(t)x_0 + C\int_0^t \Phi(t-\tau)Bu(\tau)\,d\tau + Du(t) \]

\[ \begin{align*}
\text{Total Response} & = \text{Zero Input Response} + \text{Zero State Response} \\
\end{align*} \]

\[ u(t) = 0 \forall t \Rightarrow y(t) = ZIR \quad \text{Linear function of initial state} \]

\[ x_0 = 0 \Rightarrow y(t) = ZSR \quad \begin{align*}
& \cdot \text{Linear function of input} \\
& \cdot \text{cf. linear system definition in SS1} \\
& \cdot \text{Convolution integral} \\
\end{align*} \]
Zero input transition

• If we know the state transition matrix we can (in principle) compute all solutions of linear system
• Given matrix $A$ would like to compute

$$\Phi(t) = e^{At} = I + At + \ldots + \frac{A^k t^k}{k!} + \ldots$$

• Many ways of doing this
  – Summing infinite series (in some rare cases!)
  – Using eigenvalues and eigenvectors (this set of notes)
  – Using the Laplace transform (later)
  – Numerically (later)
• Using eigenvalues at least two methods
  – Using Cayley Hamilton Theorem (Theorem 2.1)
  – Using eigenvalue decomposition (used here)
E-values and E-vectors: Rough idea

- Recall that (p. 2.15) $A w_i = \lambda_i w_i$
- ZIT
  \[
  \dot{x}(t) = Ax(t) \implies x(t) = \Phi(t)x(0)
  \]
  \[
  x(0) = w_i \implies \dot{x}(0) = Aw_i = \lambda_i w_i
  \]
- i.e. if we start on e-vector we stay on e-vector
- $\|x(t)\|$ increases/decreases depending on sign of $\lambda$
- E.g.

\[
\begin{align*}
  n &= 2, \lambda_1 < 0, \lambda_2 > 0 \\
\end{align*}
\]
Transition matrix computation

- Change of coordinates using matrix of eigenvectors
- Assume **matrix diagonalizable** (p.2.17)

\[ AW = W\Lambda \Rightarrow A = W\Lambda W^{-1} \]

- Therefore (Fact 2.18)

\[ \Phi(t) = e^{At} = We^{\Lambda t}W^{-1} \]

where

\[ e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \]

**Exercise:** Prove this

**Exercise:** Prove by induction that

\[ A^k = W\Lambda^k W^{-1} \]
Example: RLC circuit

- Recall that (p. 1.14)

\[
\frac{d}{dt} \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} v_s(t)
\]

- Set \( R=3, C=0.5, L=1 \)

\[
A = \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix}
\]

- Eigenvalues:

\( \lambda_1 = -1, \lambda_2 = -2 \)

- Eigenvectors:

\[
w_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, w_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]
RLC circuit: Transition matrix

\[ e^{At} = W e^{\Lambda t} W^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \]

\[ \Phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & 2e^{-t} - 2e^{-2t} \\ -e^{-t} + e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \]

Using Matlab:
>> A=[0 2;-1 -3];
>> [W,L]=eig(A)

\[ W = \begin{bmatrix} 0.8944 & -0.7071 \\ -0.4472 & 0.7071 \end{bmatrix} \]

\[ L = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \]
Because $w_1$ and $w_2$ linearly independent they form a basis for $\mathbb{R}^2$ (p. 2.8).

Therefore all initial conditions can be written as

$$x(0) = a_1 w_1 + a_2 w_2$$

Therefore ZIT $\to 0$ for all initial conditions.
RLC circuit: Step input

ZST with \( u(t) = V \), for \( t \geq 0 \)

\[
x(t) = \int_{0}^{t} \Phi(t-\tau) Bu(\tau) \, d\tau \\
= \Phi(t) \int_{0}^{t} \Phi(-\tau) B V \, d\tau \\
= \left[ -2e^{-t} + e^{-2t} + 1 \right] V \quad \xrightarrow[t \to \infty]{V} \left[ \begin{array}{c} V \\ 0 \end{array} \right]
\]
Notes: Diagonalizable matrices

- For **diagonalizable matrices**, state transition matrix linear combination of terms of the form $e^{\lambda_i t}$
- Generally $\lambda_i = \sigma \pm j\omega \in \mathbb{C}, \sigma, \omega \in \mathbb{R}$
- So ZIT linear combination of terms of the form
  - 1 if $\lambda_i = 0$  \( (\sigma = 0, \omega = 0) \)
  - $e^{\sigma t}$ if $\lambda_i = \sigma$  \( (\sigma \neq 0, \omega = 0) \)
  - $\sin(\omega t)$ and $\cos(\omega t)$ if $\lambda_i = \pm j\omega$  \( (\sigma = 0, \omega \neq 0) \)
  - $e^{\sigma t} \sin(\omega t)$ and $e^{\sigma t} \cos(\omega t)$ if $\lambda_i = \sigma \pm j\omega$  \( (\sigma \neq 0, \omega \neq 0) \)
- Part of ZIT corresponding to $\lambda_i = \sigma \pm j\omega$
  - Constant if $\sigma = 0, \omega = 0$
  - Converges to 0 if $\sigma < 0$
  - Periodic if $\sigma = 0, \omega \neq 0$
  - Goes to infinity if $\sigma > 0$
Typical ZIT for diagonalizable matrices
Stability: Zero input transition

- Consider the system \( \dot{x}(t) = Ax(t) + Bu(t) \)
- Let \( x(t) \) be its ZIT \( x(t) = \Phi(t)x_0 = e^{At}x_0 \)

**Definition:** The system is called **stable** if for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
\text{if } \|x_0\| \leq \delta \text{ then } \|x(t)\| \leq \varepsilon \text{ for all } t \geq 0.
\]

Otherwise the system is called **unstable.**

- i.e. if the state starts small it stays small (p. 2.4)
- or you can keep the state as close as you want to 0 by starting close enough
Asymptotic stability

**Definition:** The system is called *asymptotically stable* if it is stable and in addition

\[ \|x(t)\| \to 0 \text{ as } t \to \infty \]

- i.e. not only do we stay close to 0 but also converge to it

**Exercise:** Show that \( \|x(t)\| \to 0 \) if and only if \( x(t) \to 0 \)

- Note that
  - Definitions do not require diagonalizable matrices
  - In fact we, will see that they also work for nonlinear systems (Notes 7)
Diagonalizable matrices

**Theorem 3.1**: System with diagonalizable $A$ matrix is:
- Stable if and only if $\text{Re}[\lambda_i] \leq 0, \forall i$
- Asymptotically stable if and only if $\text{Re}[\lambda_i] < 0, \forall i$
- Unstable if and only if $\exists i : \text{Re}[\lambda_i] > 0$

- **Proof**: By inspection!
- Transition matrix linear combination of $e^{\lambda_it}$
  - $\text{Re}[\lambda_i] < 0, \forall i \Rightarrow$ for all initial conditions ZIT tends to zero
  - $\text{Re}[\lambda_i] \leq 0, \forall i \Rightarrow$ ZIT remains bounded and for some initial conditions is periodic (or constant)
  - $\exists i : \text{Re}[\lambda_i] > 0 \Rightarrow$ for some initial conditions ZIT tends to infinity
Phase plane plots

- For two state variables \((n=2)\)
- \(x_2(t)\) vs \(x_1(t)\) parameterized by \(t\) for different initial states

Trajectory plot
\[x_2(t)/x_1(t)\] vs \(t\)

Phase plane plot
\[x_2(t)\] vs \(x_1(t)\)

Stable node
\[w_1\]
\[w_2\]
Phase plane plots

- **Saddle point**
  - $\lambda_1 > 0, \lambda_2 < 0$
  - $w_1$
  - $w_2$

- **Unstable node**
  - $\lambda_1, \lambda_2 > 0$
  - $w_1$
  - $w_2$
Phase plane plots

\[ \lambda_1 = \sigma + j\omega, \lambda_2 = \sigma - j\omega; \quad \sigma < 0 \]

**Stable focus**

\[ \lambda_1 = \sigma + j\omega, \lambda_2 = \sigma - j\omega; \quad \sigma < 0 \]

**Center**

\[ \lambda_1 = \pm j\omega, \lambda_2 = \pm j\omega \]
Phase plane plots

\[ \lambda_1 = \sigma + j\omega, \lambda_2 = \sigma - j\omega; \sigma > 0 \]

Unstable focus

\[ \lambda_1 = \sigma + j\omega, \lambda_2 = \sigma - j\omega; \sigma > 0 \]
Non-diagonalizable matrices (examples)

\[ A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow e^{A_1t} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \]

**Exercise:** What are the eigenvalues of \( A_1 \) and \( A_2 \)?
What are the eigenvectors?
What goes wrong with their diagonalization?

\[ A_2 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \Rightarrow e^{A_2t} = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix} \]

**Exercise:** Prove the formulas for the transition matrices

**Exercise:** Repeat the computations for the matrices

\[ A_3 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \ A_4 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \]
Non-diagonalizable matrices (general)

- Distinct e-values $\rightarrow$ matrix diagonalizable (Fact 2.18)
- Assume some e-value repeated $r$ times, $\lambda_1 = \lambda_2 = \ldots = \lambda_r = \sigma \pm j\omega$
- In general, ZIT linear combination of terms of the form
  
  $- 1, t, t^2, \ldots, t^{r-1}$ if $\sigma = 0, \omega = 0$
  $- e^{\sigma t}, te^{\sigma t}, \ldots, t^{r-1}e^{\sigma t}$ if $\sigma \neq 0, \omega = 0$
  $- \sin(\omega t), \cos(\omega t), t \sin(\omega t), \ldots, t^{r-1} \cos(\omega t)$ if $\sigma = 0, \omega \neq 0$
  $- e^{\sigma t} \sin(\omega t), e^{\sigma t} \cos(\omega t), \ldots, t^{r-1}e^{\sigma t} \cos(\omega t)$ if $\sigma \neq 0, \omega \neq 0$
- Can be shown using generalized eigenvectors and Jordan canonical form
Non-diagonalizable matrices (general)

Note that:

• If $\sigma < 0$,
  - $t^k e^{\sigma t}, t^k e^{\sigma t} \cos \omega t, t^k e^{\sigma t} \sin \omega t \xrightarrow{t \to \infty} 0$
  - Hence ZIT tends to zero (for some initial states)

• If $\sigma > 0$,
  - $t^k e^{\sigma t}, t^k e^{\sigma t} \cos \omega t, t^k e^{\sigma t} \sin \omega t \xrightarrow{t \to \infty} \infty$
  - Hence ZIT tends to infinity (for some initial states)

• If $\sigma = 0$,
  - $1, \cos \omega t, \sin \omega t$ remain bounded
  - $t^k, t^k \cos \omega t, t^k \sin \omega t \xrightarrow{t \to \infty} \infty$ for $k \geq 1$
  - ZIT may remain bounded or tend to infinity
  - Cannot tell just by looking at e-values
Non-diagonalizable matrices: Stability

**Theorem 3.2:** The system is:
- Asymptotically stable if and only if $\text{Re}[\lambda_i] < 0 \ \forall i$
- Unstable if $\exists i: \text{Re}[\lambda_i] > 0$

- Subtle differences with diagonalizable case
  - Asymptotic stability condition the same
  - Instability condition sufficient but not necessary

- Reason is that if $\forall i \ \text{Re}[\lambda_i] \leq 0$ but $\exists i \ \text{Re}[\lambda_i] = 0$ then stability not determined by e-values alone.
  - ZIT may remain bounded for all initial conditions
  - ZIT may go to infinity for some initial conditions
  - If matrix non-diagonalizable, but no e-value with $\text{Re}[\lambda_i] = 0$ is repeated then Theorem 3.1 still applies
Zero state transition: Dirac function

- Can be thought of as a function of time which is
  - Infinite at $t=0$
  - Zero everywhere else
  - Satisfies $\int_{-\infty}^{\infty} \delta(t) = 1$

- Can be thought of as the limit as $\varepsilon \to 0$ of (among others)

\[
\delta(t) = \begin{cases} 
\infty & \text{if } t = 0 \\
0 & \text{if } t \neq 0 
\end{cases}
\]

\[
\delta_\varepsilon(t) = \begin{cases} 
0 & \text{if } t < 0 \\
\frac{1}{\varepsilon} & \text{if } 0 \leq t < \varepsilon \\
0 & \text{if } t \geq \varepsilon 
\end{cases}
\]
**Impulse transition** \( h(t) \) \((n=m=1)\)

- \( n = m = 1 \Rightarrow x(t) \in \mathbb{R}, u(t) \in \mathbb{R} \)

\[
\dot{x}(t) = ax(t) + bu(t), \quad a \in \mathbb{R}, b \in \mathbb{R}
\]

- State impulse transition \( h(t) \) is ZST \((x_0 = 0)\) for \( u(t) = \delta(t) \)

\[
h(t) = \int_0^t \Phi(t - \tau)B\delta(\tau)d\tau
\]

\[
= e^{at}\int_0^t e^{-a\tau} b\delta(\tau) d\tau = e^{at} b
\]

**Exercise:** Show that impulse transition also ZIT for appropriate \( x_0 \)

- General ZST convolution of impulse transition with input

\[
x(t) = \int_0^t \Phi(t - \tau)Bu(\tau)d\tau = \int_0^t e^{a(t-\tau)}bu(\tau)d\tau
\]

\[
= \int_0^t h(t - \tau)u(\tau) d\tau = (h \ast u)(t)
\]
Example: RC circuit (p. 3.11)

• For the RC circuit: \( A = -\frac{1}{RC} \in \mathbb{R}, B = \frac{1}{RC} \in \mathbb{R} \)

\[
\Phi(t) = e^{-\frac{t}{RC}}
\]

• Impulse transition

\[
h(t) = \Phi(t)b = \frac{1}{RC}e^{-\frac{t}{RC}}
\]

• Unit step response: ZST with

\[
u(t) = \begin{cases} 
1 & t \geq 0 \\
0 & t < 0 
\end{cases} \quad \Rightarrow x(t) = \int_{0}^{t} h(t - \tau) \cdot 1 \cdot d\tau = \left( 1 - e^{-\frac{t}{RC}} \right)
\]
Impulse transition $H(t)$ (general)

- For general $n, m$ impulse transition given by matrix

$$H(t) = \begin{bmatrix}
    h_{11}(t) & \ldots & h_{1m}(t) \\
    \vdots & \ddots & \vdots \\
    h_{n1}(t) & \ldots & h_{nm}(t)
\end{bmatrix} = \Phi(t)B \in \mathbb{R}^{n \times m}$$

- $h_{ij}(t)$ equal to $x_i(t)$ when
  - $x(0) = 0$
  - $u_j(t) = \delta(t), u_k(t) = 0 \quad k \neq j$

- Again, ZST convolution of input with impulse transition

$$x(t) = (H \ast u)(t)$$

Integral computed element by element
Output impulse response $K(t)$

- Usually interested in input-output behavior
- Output impulse response: output solution to
  - Input $\delta(t)$
  - Initial state $x(0) = 0$
- Combine impulse transition formula and output map, output impulse response given by

$$K(t) = C\Phi(t)B + D\delta(t) \in \mathbb{R}^{p \times m}$$

and output ZSR to input $u(t)$ is

$$y(t) = (K \ast u)(t)$$

**Exercise:** Verify this using the formula on p. 3.8
Stability with inputs: Zero state transition

• Consider the system \( \dot{x}(t) = Ax(t) + Bu(t) \)
• Zero state transition \( x(t) = \int_0^t \Phi(t - \tau)Bu(\tau) \, d\tau \)

**Theorem 3.3:** Assume that \( \text{Re}[\lambda_i] < 0 \quad \forall i \). Then there exists \( \alpha \geq 0 \) such that ZST, \( x(t) \), satisfies
\[
\|u(t)\| \leq M \quad \forall t \geq 0 \quad \Rightarrow \quad \|x(t)\| \leq \alpha M \quad \forall t \geq 0
\]
If in addition \( u(t) \xrightarrow{t \to \infty} 0 \) then \( x(t) \xrightarrow{t \to \infty} 0 \).

• So small inputs lead to small states. If in addition the input goes to zero, so does the state
• Asymptotic stability needed, \( \text{Re}[\lambda_i] \leq 0 \) not enough
**Stability with inputs: Full response**

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

- Complete response = ZIR + ZSR
- If \( \text{Re}[\lambda_i] < 0 \ \forall i \)
  - ZIT/ZIR and ZST/ZSR small if input and \( x(0) \) small
  - Bounded input, bounded state (BIBS) property
  - Bounded input, bounded output (BIBO) property
  - ZIT/ZIR and ZST/ZSR tend to 0 if input tends to 0
  - Hence output tends to 0 if input tends to 0
Signal- und Systemtheorie II
D-ITET, Semester 4

Notes 4: Energy, Controllability, Observability

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Wien oscillator: Circuit

\[ i_{0}(t) = 0 \Rightarrow i_{1}(t) = i_{0}(t) \]

\[
\begin{align*}
    i_{1}(t) &= \frac{v_{C_{2}}(t)}{R} \\
    i_{0}(t) &= \frac{v_{C_{2}}(t) - v_{0}(t)}{(k-1)R} \\
    v_{0}(t) &= k v_{C_{2}}(t)
\end{align*}
\]

\[ v_{in} = 0 \Rightarrow v_{R}(t) = v_{C_{2}}(t) \Rightarrow \]
**Wien oscillator: State equations**

- Linear circuit
- State Variables: $v_{c_1}(t), v_{c_2}(t)$

$$x(t) = \begin{bmatrix} v_{c_1}(t) \\ v_{c_2}(t) \end{bmatrix} \in \mathbb{R}^2$$

- Input Variable: none (autonomous)

$$\frac{dx(t)}{dt} = A \begin{bmatrix} v_{c_1}(t) \\ v_{c_2}(t) \end{bmatrix}$$

- KCL:
  $$\frac{v_{c_2}(t)}{R_2} + C_1 \frac{dv_{c_1}(t)}{dt} + C_2 \frac{dv_{c_2}(t)}{dt} = 0$$

- KVL:
  $$v_{c_2}(t) - v_{c_1}(t) - R_1 C_1 \frac{dv_{c_1}(t)}{dt} - k v_{c_2}(t) = 0$$
Wien oscillator: Response

- For simplicity set \( R_1 = R_2 = R, C_1 = C_2 = C \)
- Autonomous system (ZIT)
- Stability determined by sign of the real part of eigenvalues
- Eigenvalues are the roots of the characteristic polynomial

\[
\text{det}(\lambda I - A) = 0 \Rightarrow \lambda^2 + \frac{3-k}{RC} \lambda + \frac{1}{(RC)^2} = 0
\]

- For second order polynomials

\[
a \lambda^2 + b \lambda + c = 0
\]
the sign of real part of roots determined by signs of \( a, b, c \)
- This is NOT true for higher order polynomials, we need Hurwitz test
Wien oscillator: Stability

\[ \forall i, \text{Re} \left[ \lambda_i \right] < 0 \Leftrightarrow \exists i, \text{Re} \left[ \lambda_i \right] > 0 \]

- Asymp. stable
- Unstable

Exercise: Prove this

- \( a, b, c \) same sign
- \( a, b, c \) not same sign

\[ \Rightarrow \text{Degenerate case} \]

- \( k < 3 \Leftrightarrow \forall i, \text{Re} \left[ \lambda_i \right] < 0 \Rightarrow \text{Response goes to 0} \]

- \( k = 3 \Leftrightarrow \lambda_i = \pm \frac{j}{RC} \Leftrightarrow \text{Response oscillates with frequency } \omega=1/RC \)

- \( k > 3 \Leftrightarrow \text{Re} \left[ \lambda_i \right] > 0 \Rightarrow \text{Response goes to infinity (generally)} \)

\[ [75x40 to 159x61] \]

\[ [86x499] \]
Wien oscillator: Eigenvalue locus

- Eigenvalues are $\lambda = \frac{k - 3 \pm \sqrt{(k-1)(k-5)}}{2RC}$

- Real and negative $0 < k \leq 1$
- Complex, negative real part $1 < k < 3$
- Imaginary $k = 3$
- Complex, positive real part $3 < k < 5$
- Real and positive $k \geq 5$
- Roots real and equal $k = 1$ or $5$ (critical damping)

Exercise: Show this

Exercise: Simulate the Wien oscillator for $k = 0.5, 2, 3, 4, 6$ and plot $x_1$ vs. $x_2$

Exercise: Plot locus of the e-values as $k$ goes from $0$ to infinity (in matlab)
System energy

- For the Wien oscillator
  \[ E(t) = \frac{1}{2} C_1 v_{c_1}^2(t) + \frac{1}{2} C_2 v_{c_2}^2(t) = \frac{1}{2} x(t)^T \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} x(t) \]
- Quadratic function of the state \( E(t) = \frac{1}{2} x(t)^T Q x(t) \)
- Matrix \( Q \)
  - Symmetric \((Q=Q^T)\) (in this case diagonal)
  - Positive definite \( Q > 0 \), i.e. \( x^T Q x > 0 \) \( \forall x \neq 0 \)
- Any quadratic with \( Q \) that satisfies these properties serves as an “energy like” function
- Example: Coordinate change
  \( \hat{x}(t) = Tx(t) \) for \( T \) invertible

**Exercise:** Find \( \hat{Q} \) such that
\[ E(t) = \frac{1}{2} \hat{x}(t)^T \hat{Q} \hat{x}(t) \]
System power

- Instantaneous change in energy

\[ P(t) = \frac{dE(t)}{dt} = \frac{\dot{x}^T(t)Qx(t)}{2} + \frac{x^T(t)Q\dot{x}(t)}{2} \]

\[ = \frac{\left(x(t)^T A^T + u(t)^T B^T\right)Qx(t)}{2} + \frac{x(t)^T Q\left(Ax(t) + Bu(t)\right)}{2} \]

\[ = \frac{x(t)^T \left(A^T Q + QA\right)x(t)}{2} + \frac{\left(u(t)^T B^T Qx(t) + x(t)^T QBu(t)\right)}{2} \]

- For autonomous systems or when \( u = 0 \) (ZIT)

\[ P(t) = \frac{x(t)^T \left(A^T Q + QA\right)x(t)}{2} = -\frac{x(t)^T Rx(t)}{2}, \quad \text{for } R = -\left(A^T Q + QA\right) \]
System power

- Power also a quadratic of the state
- Matrix $R$ is symmetric
- If it is positive definite $R > 0$, i.e. $x^T R x > 0 \quad \forall x \neq 0$
  then energy decreases all the time
- Natural to assume that in this case system is stable

**Exercise:** Show $R$ is symmetric

**Exercise:** Compute power for the Wien oscillator.
For which values of $k$ is $R$ positive definite?

**Theorem 4.1:** The eigenvalues of $A$ have negative real part if and only if for all $R = R^T > 0$ there exists a unique $Q = Q^T > 0$
  such that $A^T Q + QA = -R$
Lyapunov equation

- Lyapunov equation

\[ A^T Q + QA = -R \]

- \( A \) and \( R \) known, linear equation in unknown \( Q \)
- Can be re-written as

\[ \hat{A}q = r \]

- Because \( Q \) and \( R \) symmetric \( n(n+1)/2 \) equations in \( n(n+1)/2 \) unknowns
- Fact 2.11 \( \rightarrow \) equation has:
  - Unique solution if \( \hat{A} \) non-singular
  - Multiple solutions or no solutions if \( \hat{A} \) singular
Lypunov functions

• Linear version of Lyapunov Theorem (Notes 7)
• Possible to solve $A^T Q + QA = -R$ efficiently (e.g. Matlab)
• For any $R = R^T > 0$ solving for Lyapunov equation for unknown $Q$ allows us to determine stability of $\dot{x}(t) = Ax(t)$
  – Unique positive definite solution $\rightarrow$ Asymptotically stable
  – No solution, multiple solutions $\rightarrow$ Not asymptotically stable
  – Non-positive definite solution

• Resulting energy-like function $V : \mathbb{R}^n \rightarrow \mathbb{R}$

\[
V(x) = \frac{1}{2} x^T Q x
\]

known as Lyapunov function

Exercise: Why is Lyapunov equation linear? Is Lyapunov function linear?
Input-State-Output relations

- Investigate the effect of
  - Input on state
  - State on output
- Two fundamental questions
  1. Can I use inputs to “drive” state to desired value
  2. Can I infer what the state is by looking at output
- Answer to 1. \( \rightarrow \) Controllability
- Answer to 2. \( \rightarrow \) Observability
- Answers hidden in structure of matrices \( A, B, \) and \( C \)
Controllability

• Consider a linear system

\[
\frac{dx}{dt}(t) = Ax(t) + Bu(t)
\]
\[
y(t) = Cx(t) + Du(t)
\]

**Definition:** The system is called **controllable** over \([0, t]\) if for all \(x(0) = x_0 \in \mathbb{R}^n\) initial conditions and all terminal \(x_1 \in \mathbb{R}^n\) conditions there exists an input
\[
u(\cdot): [0, t] \rightarrow \mathbb{R}^m\] such that \(x(t) = x_1\)
**Observations**

- In other words: For any $x_0, x_1$ we can find $u(\cdot) : [0, t] \rightarrow \mathbb{R}^m$ such that

$$x_1 = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

- Observations:
  - Input can drive the state from $x_0$ to $x_1$ in time $t$, but not necessarily keep it at $x_1$ afterwards
  - Input to state relation, outputs play no role and can be ignored for the time being
  - Definition generalizes to other systems (nonlinear, time varying, etc.) but more care is needed
Observations

**Fact 4.1:** The system is controllable over $[0, t]$ if and only if for all $x_1 \in \mathbb{R}^n$ there exists an input $u(\bullet) : [0, t] \rightarrow \mathbb{R}^m$ such that $x(t) = x_1$ starting at $x(0) = 0$

**Proof:**  
**Exercise:** Prove “only if” part  
If: To drive the system from $x(0) = x_0$ to $x(t) = x_1$ use input that drives it from $\tilde{x}(0) = 0$ to $\tilde{x}(t) = x_1 - e^{At}x_0$

**Fact 4.2:** The system is controllable over $[0, t]$ if and only if for all $x_0 \in \mathbb{R}^n$ there exists an input $u(\bullet) : [0, t] \rightarrow \mathbb{R}^m$ such that $x(t) = 0$ starting at $x(0) = x_0$

**Proof:**  
**Exercise:** Prove “only if” part  
If: To drive the system from $x(0) = x_0$ to $x(t) = x_1$ use input that drives it from $\tilde{x}(0) = x_0 - e^{-At}x_1$ to $\tilde{x}(t) = 0$
Controllability gramian

Given time $t$, define controllability gramian

$$W_C(t) = \int_{0}^{t} e^{A\tau} BB^T e^{A^T\tau} d\tau \in \mathbb{R}^{n \times n}$$

Exercise: Show that $W_C(t) = W_C^T(t) \geq 0$

**Fact 4.3:** The system is controllable over $[0, t]$ if and only if $W_C(t)$ is invertible

**Proof: If.** Drive system from $x_0 \in \mathbb{R}^n$ to $x_1 \in \mathbb{R}^n$ in time $t$.

By Fact 4.1, assume $x_0=0$ and select

$$u(\tau) = B^T e^{A^T(t-\tau)}W_C(t)^{-1}x_1, \quad \tau \in [0, t]$$

Exercise: Complete the “if” part
Controllability gramian

Only if: If \( W_C(t) \) is not invertible then (Fact 2.12, 2.17) there exists \( z \in \mathbb{R}^n \) with \( z \neq 0 \) such that

\[
W_C(t)z = 0 \iff z^T W_C(t)z = 0 \iff \int_0^t z^T e^{A\tau} BB^T e^{A^T\tau} z \, d\tau = 0
\]

\[
\iff \int_0^t \left\| z^T e^{A\tau} B \right\|^2 \, d\tau = 0 \iff z^T e^{A\tau} B = 0 \quad \text{for all } \tau \in [0,t]
\]

Therefore \( z^T x(t) = \int_0^t z^T e^{A(t-\tau)} Bu(\tau) \, d\tau = 0 \)

and only \( x(t) \) orthogonal to \( z \) can be reached from \( x(0)=0 \). By Fact 4.1.1, the system is not controllable.
Controllability test

Define the controllability matrix

\[ P = [B \ AB \ A^2B \cdots A^{n-1}B] \in \mathbb{R}^{n \times n}m \]

**Theorem 4.2:** The system is controllable over \([0, t]\) if and only if the rank of \(P\) is \(n\)

The rank of \(P\) is at most \(n\) since it has \(n\) rows (Fact 2.5)

**Proof:** We know that the system is controllable over \([0, t]\) if and only if \(W_C(t)\) is invertible. \(W_C(t)\) is invertible if and only if for \(z \in \mathbb{R}^n\)

\[ W_C(t)z=0 \Leftrightarrow z=0 \]

(else \(W_C(t)\) has 0 as an eigenvalue, Fact 2.17). If we can show

\[ W_C(t)z=0 \Leftrightarrow P^Tz=0 \]

this would imply \(P^T\) has rank \(n\) if and only if \(W_C(t)\) is invertible.
Controllability test: Proof

As in the proof of Fact 4.3 we can show that
\[ W_C(t)z = 0 \iff B^T e^{A^T \tau} z = 0 \quad \text{for all } \tau \in [0, t] \]

By Taylor series, the last part holds if and only if \( B^T e^{A^T \tau} z \) and all its derivatives at \( \tau = 0 \) are equal to zero, in other words
\[
B^T e^{A^T \tau} z \bigg|_{\tau=0} = B^T z = 0 \\
\frac{d}{d\tau} B^T e^{A^T \tau} z \bigg|_{\tau=0} = B^T A^T z = 0
\]
and so on, until
\[
\frac{d^{n-1}}{d\tau^{n-1}} B^T e^{A^T \tau} z \bigg|_{\tau=0} = B^T (A^{n-1})^T z = 0
\]

Higher derivatives (involving \( A^n, A^{n+1}, \text{etc.} \)) are then automatically zero by the Cayley-Hamilton Theorem 2.1. Summarizing
\[ W_C(t)z = 0 \iff P^T z = 0 \]
and the system is controllable if and only if \( P \) has rank \( n \).
Example: OpAmp circuit

Ideal amplifier:

\[ i_1(t) = i_0(t) + i_C(t) + i_L(t) \]

\[ \Rightarrow \frac{v_{in}(t)}{R_1} = \frac{v_C(t)}{R_0} + C \frac{dv_C(t)}{dt} + i_L(t) \]

\[ \Rightarrow \frac{dv_C(t)}{dt} = - \frac{1}{C} i_L(t) - \frac{1}{R_0 C} v_C(t) + \frac{1}{R_1 C} v_{in}(t) \]

\[ L \frac{di_L(t)}{dt} = v_C(t) \Rightarrow \frac{di_L(t)}{dt} = \frac{v_C(t)}{L} \]

\[ v_0(t) = -v_C(t) \]
Example: OpAmp circuit

The state space matrixes are:

\[
A = \begin{bmatrix}
0 & 1/L \\
-1/C & -1/R_0C
\end{bmatrix},
B = \begin{bmatrix}
0 \\
1/R_1C
\end{bmatrix},
C = \begin{bmatrix}
0 & 1
\end{bmatrix}
\]

Note that the input affects only one of the two states directly. It can use this state to influence the second state through the dynamics encoded in the matrix \( A \).

\[
P = \begin{bmatrix}
0 & \frac{1}{R_1LC} \\
\frac{1}{R_1C} & -\frac{1}{R_0R_1C^2}
\end{bmatrix}
\]

\[\implies \det(P) = \frac{-1}{R_1^2LC^2} \neq 0\]
Observations

- Easy test for controllability
- Requires matrix multiplications and rank test, instead of integration of matrix exponential
- Proof of Theorem 4.2 implies the following

**Corollary 4.1**: The set of $x_1 \in \mathbb{R}^n$ for which $\exists u(\cdot) : [0, t] \to \mathbb{R}^m$ such that $x(t) = x_1$ starting at $x(0) = 0$ is equal to $\text{Range}(P)$

**Fact 4.4**: $W_C(t)$ is invertible for some $t > 0$ if and only if it is invertible for all $t > 0$.

- Roughly speaking, if the system is controllable can get from any state to any other state as fast as we like
- The faster we go, the more the “energy” and the bigger the inputs we will need

**Exercise**: Prove Fact 4.4
Minimum energy inputs

Consider as the “energy” of the input the quantity

\[ \int_0^t u(\tau)^T u(\tau) d\tau = \int_0^t \|u(\tau)\|^2 d\tau \]

In our example of p. 4.20

\[ \int_0^t \|u(\tau)\|^2 d\tau = R_1 \int_0^t v_{in}(\tau)i_1(\tau) d\tau = R_1 \cdot (\text{energy provided by } v_{in}) \]

**Theorem 4.2:** Assume that the system is controllable. Given \( x_1 \in \mathbb{R}^n \) and \( t > 0 \), the input that drives the system from \( x(0) = 0 \) to \( x(t) = x_1 \) and has the minimum energy is given by

\[ u_m(\tau) = B^T e^{A^T(t-\tau)} W_C(t)^{-1} x_1, \quad \text{for } \tau \in [0, t] \]
Minimum energy inputs: Proof

Proof: In the proof of Fact 4.3 we saw that the proposed $u_m(.)$ drives the system from $x(0)=0$ to $x(t)=x_1$. Its energy is

$$\int_0^t u_m(\tau)^T u_m(\tau) d\tau = \int_0^t x_1^T W_C(t)^{-1} e^{A(t-\tau)} BB^T e^{A^T(t-\tau)} W_C(t)^{-1} x_1 d\tau$$

$$= x_1^T W_C(t)^{-1} \left( \int_0^t e^{A(t-\tau)} BB^T e^{A^T(t-\tau)} d\tau \right) W_C(t)^{-1} x_1 = x_1^T W_C(t)^{-1} x_1$$

To show that this energy is minimum, consider any other input $u(.)$ that drives the state from $x(0)=0$ to $x(t)=x_1$. $u(.)$ can be written as $u(\tau) = u_m(\tau) + \hat{u}(\tau)$. So its energy will be

$$\int_0^t u(\tau)^T u(\tau) d\tau = \int_0^t (u_m(\tau) + \hat{u}(\tau))^T (u_m(\tau) + \hat{u}(\tau)) d\tau$$

$$= \int_0^t (u_m(\tau)^T u_m(\tau) + u_m(\tau)^T \hat{u}(\tau) + \hat{u}(\tau)^T u_m(\tau) + \hat{u}(\tau)^T \hat{u}(\tau)) d\tau$$
Minimum energy inputs: Proof

But \( x(t) = \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau = \int_0^t e^{A(t-\tau)} B(u_m(\tau) + \hat{u}(\tau)) d\tau \)

\[ = \int_0^t e^{A(t-\tau)} Bu_m(\tau) d\tau + \int_0^t e^{A(t-\tau)} B\hat{u}(\tau) d\tau = x_1 + \int_0^t e^{A(t-\tau)} B\hat{u}(\tau) d\tau \]

Since \( x(t) = x_1 \), we have that \( \int_0^t e^{A(t-\tau)} B\hat{u}(\tau) d\tau = 0 \)

and \( \int_0^t u_m(\tau)^T \hat{u}(\tau) d\tau = \int_0^t \hat{u}(\tau)^T u_m(\tau) d\tau = \int_0^t x_1^T W_C(t)^{-1} e^{A(t-\tau)} B\hat{u}(\tau) d\tau = 0 \)

Therefore

\[ \int_0^t u(\tau)^T u(\tau) d\tau = x_1^T W_C(t)^{-1} x_1 + \int_0^t \hat{u}(\tau)^T \hat{u}(\tau) d\tau \geq \int_0^t u_m(\tau)^T u_m(\tau) d\tau \]
Observability

\[ \frac{dx}{dt}(t) = Ax(t) + Bu(t) \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p \]

\[ y(t) = Cx(t) + Du(t) \]

**Definition**: The system is called **observable** over \([0, t]\) if given \(u(\cdot) : [0, t] \to \mathbb{R}^m\) and \(y(\cdot) : [0, t] \to \mathbb{R}^p\) we can uniquely determine the value of \(x(\cdot) : [0, t] \to \mathbb{R}^n\)

- Again time of observation, \(t\), will turn out to play no role
- Inputs play little role, just carried along
- Generalizations (to e.g. non-linear systems) possible, but care is needed
Initial state observability

- Recall that $x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$

- Therefore to infer $x(\cdot) : [0, t] \rightarrow \mathbb{R}^n$ given $u(\cdot) : [0, t] \rightarrow \mathbb{R}^m$
  it suffices to infer the initial condition $x(0) = x_0$

- Assume that two initial conditions, $x_0$ and $\hat{x}_0$, under the same input $u(\cdot) : [0, t] \rightarrow \mathbb{R}^m$ lead to the same output $y(\cdot) : [0, t] \rightarrow \mathbb{R}^p$, i.e. $\forall \tau \in [0, t]$

$$Ce^{At} x_0 + C\int_0^\tau e^{A(\tau-s)} Bu(s) ds + Du(\tau) = Ce^{At} \hat{x}_0 + C\int_0^\tau e^{A(\tau-s)} Bu(s) ds + Du(\tau)$$

- Then $Ce^{At} (x_0 - \hat{x}_0) = 0$, for all $\tau \in [0, t]$

- $x \in \mathbb{R}^n$ such that $Ce^{At} x = 0 \ \forall \tau \in [0, t]$ called **unobservable**
Unobservable states

- $x \in \mathbb{R}^n$ unobservable if and only if $Ce^{A\tau}x = 0$ for all $\tau \in [0, t]$
- Note that if $x=0$ then $Ce^{A\tau}x = 0$, so $x=0$ is an unobservable state
- System is observable if and only if $x=0$ is the only unobservable state
- In this case the initial state $x_0$ is uniquely determined by the zero input response since

\[
Ce^{A\tau}x_0 = Ce^{A\tau}\hat{x}_0 \text{ for all } \tau \in [0, t] \\
\iff Ce^{A\tau}(x_0 - \hat{x}_0) = 0 \text{ for all } \tau \in [0, t] \\
\iff (x_0 - \hat{x}_0) \text{ unobservable} \\
\implies (x_0 - \hat{x}_0) = 0 \implies x_0 = \hat{x}_0
\]

Exercise: Show that the unobservable states form a subspace
Observability

• Note that two initial conditions that under the same input lead to the same output differ by an unobservable state

• By a Taylor series argument \( Ce^{A\tau} x = 0 \) for all \( \tau \in [0, t] \) if and only if all its derivatives at \( \tau = 0 \) equal to 0

\[
Ce^{A\tau} x \bigg|_{\tau=0} = Cx = 0, \quad \frac{d}{dt} Ce^{A\tau} x \bigg|_{\tau=0} = CAx = 0, \ldots
\]

\[
\frac{d^{n-1}}{d\tau^{n-1}} Ce^{A\tau} x \bigg|_{\tau=0} = CA^{n-1} x = 0
\]

• By the Cayley Hamilton Theorem 2.1, if \( CA^k x = 0 \) for \( k=0, 1, \ldots, n-1 \) then \( CA^k x = 0 \) for all \( k>n-1 \)

**Exercise:** Show this
Observability

- Therefore a state $x$ is unobservable if and only if $Qx = 0$

\[
Q = \begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{bmatrix} \in \mathbb{R}^{np \times n}
\]

- If $Q$ is full rank then the only unobservable state is 0
- In this case, system is observable since

\[
Ce^{At}(x_0 - \hat{x}_0) = 0 \text{ for all } \tau \in [0,t] \iff x_0 = \hat{x}_0
\]

**Theorem 4.3:** The system is observable over $[0, t]$ if and only if the rank of the matrix $Q$ is $n$
Example: OpAmp circuit (p. 4.20)

- \[ Q = \begin{bmatrix} 0 & -1 \\ 1/C & 1/R_0C \end{bmatrix} \Rightarrow \det(Q) = \frac{1}{C} \neq 0 \text{ therefore it is observable} \]

- We are only measuring one of the two states directly
- We can infer the value of the other state by its effect on the measured state through the dynamics encoded in \( A \)
- Roughly speaking use measured state + all its derivatives to deduce the value of the unmeasured states
Observability Gramian

One can also construct and observability gramian

\[ W_O(t) = \int_{0}^{t} e^{A^T \tau} C^T C e^{A \tau} d\tau \in \mathbb{R}^{n \times n} \]

**Exercise:** Show that \( W_O(t) = W_O^T(t) \geq 0 \)

**Fact 4.5:** The system is observable over \([0, t]\) if and only if \( W_O(t) \) is invertible. If the system is observable over some \([0, t]\) then it is also observable over all \([0, t']\)

**Notes**
- Checking the rank of matrix \( Q \) is easier
- Rank of \( Q \) at most \( n \) (\( n \) columns)
- Time of observation is immaterial

**Corollary 4.2:** Set of unobservable states equal to \( \text{Null}(Q) \)
Output derivative interpretation

Consider differentiating \(y(t)\) along \(\dot{x}(t) = Ax(t) + Bu(t)\)

\[\begin{aligned}
y(t) &= Cx(t) + Du(t) \\
\dot{y}(t) &= C\dot{x}(t) + D\dot{u}(t) = CAx(t) + CBu(t) + D\dot{u}(t) \\
\ddot{y}(t) &= CA^2 x(t) + CABu(t) + CB\ddot{u}(t) + D\dddot{u}(t)
\end{aligned}\]

\[
\begin{bmatrix}
y(0) \\
\dot{y}(0) \\
\vdots \\
y^{(n-1)}(0)
\end{bmatrix}
= 
\begin{bmatrix}
    C & & & \\
    CA & & & \\
    \vdots & & & \\
    CA^{n-1} & & & 
\end{bmatrix}
\begin{bmatrix}
x(0) \ \\
\end{bmatrix}
+ 
\begin{bmatrix}
    D & 0 & \cdots & 0 \\
    CB & D & \cdots & 0 \\
    CAB & CB & \cdots & 0 \\
    CA^{n-2} B & CA^{n-3} B & \cdots & D
\end{bmatrix}
\begin{bmatrix}
    u(0) \\
    \ddot{u}(0) \\
    \vdots \\
    u^{(n-1)}(0)
\end{bmatrix}
\]

\[Y = Qx(0) + KU \quad Y \in \mathbb{R}^{np}, \quad K \in \mathbb{R}^{np \times nm}, \quad U \in \mathbb{R}^{nm}\]
Output derivative interpretation

\[ Y = Qx(0) + KU \]

- System of linear equations to be solved for \( x(0) \)
- If \( p=1 \), \( Q \in \mathbb{R}^{n \times n} \) has rank \( n \) \( \Rightarrow Q \) invertible
  \[ x(0) = Q^{-1}(Y - KU) \]
- If \( p>1 \), more equations than unknowns, least squares solution. If \( Q \) has rank \( n \), pseudo-inverse (Fact 2.14)
  \[ x(0) = \left( Q^T Q \right)^{-1} Q^T (Y - KU) \]
But …

- Differentiating measurements is a bad idea
- Noise gets amplified
- Intuition: Sinusoidal signal corrupted by small amplitude, high frequency noise
  \[ y(t) = a \sin(\omega t) + b \sin(\omega_n t) \quad b \ll a, \omega_n \gg \omega \]
- Signal-to-noise ratio: \( \text{SNR} = \frac{a}{b} \gg 1 \)
  \[ \dot{y}(t) = \omega a \cos(\omega t) + \omega_n b \cos(\omega_n t) \Rightarrow \text{SNR} = \frac{a\omega}{b\omega_n} \ll \frac{a}{b} \]
  \[ \ddot{y}(t) = -\omega^2 a \sin(\omega t) - \omega_n^2 b \sin(\omega_n t) \Rightarrow \text{SNR} = \frac{a\omega^2}{b\omega_n^2} \ll \frac{a\omega}{b\omega_n} \]
- Derivative of signal soon becomes useless
Observers

• Instead of differentiating, build a “filter”
• Progressively construct estimate of the state \( \tilde{x}(t) \in \mathbb{R}^n \)
• Start with some (arbitrary) initial guess \( \tilde{x}(0) \in \mathbb{R}^n \)
• Measure \( y(t) \) and \( u(t) \)
• Update estimate according to
  \[
  \frac{d\tilde{x}}{dt}(t) = A\tilde{x}(t) + Bu(t) + L \left[ y(t) - C\tilde{x}(t) - Du(t) \right]
  \]
  • Mimic evolution of true state, plus correction term
• Gain matrix \( L \)
• Error dynamics
  \[
e(t) = x(t) - \tilde{x}(t) \Rightarrow \dot{e}(t) = (A - LC)e(t)
  \]
Observers

**Theorem 4.4:** If the system is observable, then $L$ can be chosen such that eigenvalues of $(A-LC)$ have negative real parts.

- In this case error system is asymptotically stable
- Error goes to zero $e(t) \xrightarrow{t \to \infty} 0$
- State estimate converges to true state $\hat{x}(t) \xrightarrow{t \to \infty} x(t)$
- Convergence arbitrarily quick by choice of $L$
- In presence of noise “transients” may be bad
- Kalman filter: Optimal trade-off for $L$ if
  - State and measurement equations corrupted by noise
  - System linear and noises Gaussian
Kalman decomposition

There exists change of coordinates $T \in \mathbb{R}^{n \times n}$ invertible such that:

$$\hat{x}(t) = Tx(t) = \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \\ \hat{x}_3(t) \\ \hat{x}_4(t) \end{bmatrix} \leftarrow \text{controllable & observable}$$

$$\begin{bmatrix} \hat{A}_{11} & 0 & \hat{A}_{13} & 0 \\ \hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23} & \hat{A}_{24} \\ 0 & 0 & \hat{A}_{33} & 0 \\ 0 & 0 & \hat{A}_{43} & \hat{A}_{44} \end{bmatrix}$$

$$\hat{B} = TB = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}$$

$$\begin{bmatrix} \hat{C}_1 & 0 & \hat{C}_3 & 0 \end{bmatrix}$$

$$\hat{C} = CT^{-1}$$
Kalman decomposition

\[
\begin{bmatrix}
\hat{A}_{11} & 0 \\
\hat{A}_{21} & \hat{A}_{22}
\end{bmatrix},
\begin{bmatrix}
\hat{B}_1 \\
\hat{B}_2
\end{bmatrix}
\]

controllable,

\[
\begin{bmatrix}
\hat{A}_{11} & \hat{A}_{13} \\
0 & \hat{A}_{33}
\end{bmatrix},
\begin{bmatrix}
\hat{C}_1 \\
\hat{C}_3
\end{bmatrix}
\]

observable
Stabilizability and detectability

**Definition:** The system is **detectable** if all eigenvalues of \( \hat{A}_{22} \) and \( \hat{A}_{44} \) in the Kalman decomposition have negative real part.

Can design observer for observable part with overall observation error decaying to zero

\[
\begin{bmatrix}
\hat{A}_{11} & \hat{A}_{13} \\
0 & \hat{A}_{33}
\end{bmatrix}
\begin{bmatrix}
\hat{C}_1 \\
\hat{C}_3
\end{bmatrix}
\]

**Definition:** The system is **stabilizable** if all eigenvalues of \( \hat{A}_{33} \) and \( \hat{A}_{44} \) in Kalman decomposition have negative real part.

Can design controller for controllable part which ensures overall system asymptotically stable

\[
\begin{bmatrix}
\hat{A}_{11} & 0 \\
\hat{A}_{21} & \hat{A}_{22}
\end{bmatrix}
\begin{bmatrix}
\hat{B}_1 \\
\hat{B}_2
\end{bmatrix}
\]
Laplace Transform

- Convert time function \( f(t) \) to a complex variable function \( F(s) \)

\[
f : \mathbb{R} \rightarrow \mathbb{R} \quad \quad F : \mathbb{C} \rightarrow \mathbb{C}
\]

\[
f(t) \xleftrightarrow{L} F(s) \quad \quad F(s) = L\{f(t)\} = \int_{0}^{\infty} f(t)e^{-st} \, dt
\]

- Recall that we assume that \( f(t)=0 \) for all \( t<0 \) (p. 0.22)
- Can also be defined for matrix valued functions

\[
f : \mathbb{R} \rightarrow \mathbb{R}^{n \times m} \quad \quad F : \mathbb{C} \rightarrow \mathbb{C}^{n \times m}
\]

by taking the integral element by element
Laplace Transform: Properties

**Assumption:** The function $f(t)$ is such that the integral can be defined, i.e. $f(t)e^{-st} \xrightarrow{t \to \infty} 0$ “quickly enough”

- **Linearity** $L\left\{ a_1 f(t) + a_2 g(t) \right\} = a_1 F(s) + a_2 G(s)$

- **s shift** $L\left\{ e^{-at} f(t) \right\} = F(s + a)$

- **Time derivative** $L\left\{ \frac{d}{dt} f(t) \right\} = sF(s) - f(0)$

- **Convolution** $L\left\{ (f * g)(t) \right\} = F(s)G(s)$

**Exercise:** Prove these properties using the definition

Recall discussion on p. 0.22
Laplace Transform: Useful functions

A. Dirac function \( L\{\delta(t)\} = 1 \)

B. Step function \( L\{1\} = \frac{1}{s} \)

C. Exponential function \( L\{e^{-at}\} = \frac{1}{s + a} \)

D. Sinusoidal functions
   \[
   L\{\sin(\omega t)\} = \frac{\omega}{s^2 + \omega^2}
   \]
   \[
   L\{\cos(\omega t)\} = \frac{s}{s^2 + \omega^2}
   \]

**Exercise:** Prove C using the \( s \) shift property. Prove D using C. Prove B using A and the time derivative property.

Initial value theorem: \( \lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s) \)

Final value theorem: \( \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) \)

Whenever all limits exist
Inverse Laplace Transform

- Defined as an integral
- Laplace transforms of interest here will be proper, rational functions
  - Ratio of two polynomials in $s$
  - Degree of numerator less than or equal to degree of denominator
- In this case use partial fractions
- Example:

$$L^{-1}\left\{\frac{1}{s^2 + 3s + 2}\right\} = L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} = L^{-1}\left\{\frac{1}{s+1} - \frac{1}{s+2}\right\}$$

$$= L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-t} - e^{-2t}$$

Exercise: Compute the Laplace transform of:
- $f(t) = t, f(t) = t^n$,
- $f(t) = te^{-at}$,
- $f(t) = e^{-at} \cos(\omega t)$,
- $g(t) = \frac{d^2}{dt^2} f(t)$,
- $f(t) = \sin(\omega t + \theta)$
Back to LTI systems

Time domain:

\[
\frac{dx(t)}{dt} = Ax(t) + Bu(t) \\
y(t) = Cx(t) + Du(t)
\]

\[x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p, t \in \mathbb{R}\]

Take Laplace Transform

\[
L\left\{\frac{dx(t)}{dt}\right\} = L\left\{Ax(t) + Bu(t)\right\} \implies sX(s) - x(0) = AX(s) + BU(s)
\]

Laplace domain:

\[
X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s) \\
Y(s) = CX(s) + DU(s)
\]

\[X(s) \in \mathbb{C}^n, U(s) \in \mathbb{C}^m, Y(s) \in \mathbb{C}^p, s \in \mathbb{C}\]
Comparison with time domain

- Time domain solution

\[ x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau \]

- Take Laplace transform

\[
L\{x(t)\} = L\{e^{At}\}x_0 + L\left\{\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau\right\}
\]

\[ \Rightarrow X(s) = L\{e^{At}\}x_0 + L\{e^{At}\}BU(s) \]

- Comparing

\[ L\{e^{At}\} = (sI - A)^{-1} \in \mathbb{C}^{n\times n} \]
Example (p. 3.18)

For \( R=3, \ C=0.5, \ L=1 \)

\[
A = \begin{bmatrix}
0 & 2 \\
-1 & -3
\end{bmatrix}
\Rightarrow (sI - A) = \begin{bmatrix}
s & -2 \\
1 & s + 3
\end{bmatrix}
\]

Laplace transform of state transition matrix

\[
(sI - A)^{-1} = \frac{1}{s^2 + 3s + 2} \begin{bmatrix}
s + 3 & 2 \\
-1 & s
\end{bmatrix}
\]
Example: Transition Matrix

\[ \Phi(t) = L^{-1} \left\{ (sI - A)^{-1} \right\} = L^{-1} \left\{ \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s + 3 & 2 \\ -1 & s \end{bmatrix} \right\} \]

\[ = L^{-1} \left\{ \begin{bmatrix} \frac{s + 3}{s^2 + 3s + 2} & \frac{2}{s^2 + 3s + 2} \\ \frac{-1}{s^2 + 3s + 2} & \frac{s}{s^2 + 3s + 2} \end{bmatrix} \right\} \]

\[ = L^{-1} \left\{ \begin{bmatrix} \frac{2}{s + 1} - \frac{1}{s + 2} & \frac{2}{s + 1} - \frac{2}{s + 2} \\ \frac{-1}{s + 1} + \frac{1}{s + 2} & \frac{-1}{s + 1} + \frac{2}{s + 2} \end{bmatrix} \right\} \]

\[ \Phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & 2e^{-t} - 2e^{-2t} \\ -e^{-t} + e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \]

As before! (p. 3.19)
Example: Step transition (p. 3.21)

ZST with input \( v_s(t) = V \) for \( t \geq 0 \) (recall that \( v_s(t) = 0 \) for \( t < 0 \))

Laplace transform \( V_s(s) = V/s \)

\[
X(s) = (sI - A)^{-1} BV_s(s) = \begin{bmatrix} 2 \\ \frac{2}{s(s+1)(s+2)} \\ \frac{s}{s(s+1)(s+2)} \end{bmatrix} V
\]

\[
= \begin{bmatrix} \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2} \\ \frac{1}{s+1} - \frac{1}{s+2} \end{bmatrix} V
\]

No need to compute entire \((sI-A)^{-1}\), just second column

\[
x(t) = L^{-1} \{X(s)\} = \begin{bmatrix} -2e^{-t} + e^{-2t} + 1 \\ e^{-t} - e^{-2t} \end{bmatrix} V
\]
Example: Step transition

\[ X(s) = \begin{bmatrix} 
\frac{2}{s(s+1)(s+2)} \\
\frac{s}{s(s+1)(s+2)} \\
\frac{s}{s(s+1)(s+2)} 
\end{bmatrix} \]

Initial value theorem  \[ x(0) = \lim_{t \to 0} x(t) = \lim_{s \to \infty} sX(s) \]

\[ x(0) = \lim_{s \to \infty} \begin{bmatrix} 
\frac{2}{(s+1)(s+2)} \\
\frac{s}{(s+1)(s+2)} \\
\frac{s}{(s+1)(s+2)} 
\end{bmatrix} V = \begin{bmatrix} 0 \\
0 
\end{bmatrix} \] (ZST)

Final value theorem  \[ \lim_{t \to \infty} x(t) = \lim_{s \to 0} \begin{bmatrix} 
\frac{2}{(s+1)(s+2)} \\
\frac{s}{(s+1)(s+2)} \\
\frac{s}{(s+1)(s+2)} 
\end{bmatrix} \]

\[ V = \begin{bmatrix} V \\
0 
\end{bmatrix} \]
Example: Sinusoidal input

ZST with input \( v_s(t) = V \sin(t) \)

Laplace transform \( V_s(s) = \frac{V}{s^2 + 1} \)

\[
X(s) = (sI - A)^{-1} BV_s(s) = \begin{bmatrix} \frac{2}{(s^2 + 1)(s + 1)(s + 2)} \\ \frac{s}{(s^2 + 1)(s + 1)(s + 2)} \end{bmatrix} V
\]

\[
V_C(s) = X_1(s) = \left( \frac{-3s + 1}{5(s^2 + 1)} + \frac{1}{s + 1} - \frac{2}{5(s + 2)} \right)V
\]

\[
v_C(t) = \frac{3V}{5} \cos(t) + \frac{V}{5} \sin(t) + Ve^{-t} - \frac{2}{5}Ve^{-2t}
\]

External input response

Eigenvalue response
Example: Sinusoidal input

- The system is stable, so as $t \to \infty$

$$Ve^{-t} - \frac{2}{5}Ve^{-2t} \to 0$$

Transient solution

$$v_C(t) \to -\frac{3V}{5}\cos(t) + \frac{V}{5}\sin(t)$$

Steady state solution

- In general, for stable systems with sinusoidal input steady state solution is also sinusoidal with
  - Same frequency as input
  - Amplitude and phase determined by system matrices
Transfer function

- Consider ZSR $X(s) = (sI - A)^{-1}BU(s)$
  
  $$Y(s) = CX(s) + DU(s)$$

  $$Y(s) = \left(C(sI - A)^{-1}B + D\right)U(s)$$

- Transfer function

\[
G(s) = C(sI - A)^{-1}B + D \quad \in \mathbb{C}^{p \times m}
\]

- Summarizes system input-output behavior $Y(s) = G(s)U(s)$

- In the RLC example
  - If we measure $y = i_L$  
    \[
    C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 0 \Rightarrow G(s) = \frac{s}{(s + 1)(s + 2)}
    \]

  - If we measure $y = v_C$  
    \[
    C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0 \Rightarrow G(s) = \frac{2}{(s + 1)(s + 2)}
    \]
Transfer function structure

• System called
  – Single input, single output (SISO) if \( m=p=1 \)
  – Multi-input, multi-output (MIMO) if \( m \) or \( p > 1 \)

• SISO \( \Rightarrow \) \( B, C^T \) vectors of dimension \( n \), \( D \) a real number

\[
G(s) = \frac{\text{Adj}(sI - A)B}{\det(sI - A)} + D = \frac{\text{Adj}(sI - A)B + D \det(sI - A)}{\det(sI - A)} \in \mathbb{C}
\]

• All entries are rational functions of \( s \)
• For SISO systems

\[
G(s) = \frac{(s - z_1)(s - z_2)\cdots(s - z_k)}{(s - p_1)(s - p_2)\cdots(s - p_n)} \in \mathbb{C}
\]
Proper and strictly proper transfer function

- Consider SISO system described by rational \( G(s) \)
- Transfer function is called proper if
  numerator degree \( \leq \) denominator degree

**Fact 5.1:** SISO transfer functions arising from state space descriptions of LTI systems are always proper

- Transfer function is called strictly proper if
  numerator degree < denominator degree

**Fact 5.2:** SISO transfer functions arising from state space descriptions of LTI systems are strictly proper if and only if \( D=0 \)

- Input affects output only through system dynamics
Proper and strictly proper transfer function

\[ G(s) = \frac{(s - z_1)(s - z_2) \cdots (s - z_k)}{(s - p_1)(s - p_2) \cdots (s - p_n)} \in \mathbb{C} \]

- Proper \( \rightarrow k \leq n \), strictly proper \( \rightarrow k < n \)
- If \( G(s) \) is strictly proper, expanding polynomials

\[ G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n} \]

- Some \( b_i \) may be zero
- If no pole-zero cancellations the denominator is the characteristic polynomial of \( A \)
- i.e. poles are the eigenvalues of \( A \)
- For simplicity we will mostly consider SISO, strictly proper transfer functions in the rest of these notes
Transfer function and impulse response

- SISO systems, ZSR \( y(t) = C \int_{0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau + Du(t) \)

- Output impulse response: ZSR with \( u(t) = \delta(t) \)
  \[ K(t) = Ce^{At} B + D\delta(t) \quad (c.f. \text{Notes 3}) \]

- Taking Laplace transform
  \[ L\{K(t)\} = L\left\{Ce^{At} B + D\delta(t)\right\} = C(sI - A)^{-1} B + D \]

- Transfer function is Laplace transform of output impulse response

- ZSR: impulse response-input convolution \( y(t) = (K*u)(t) \)

\[ Y(s) = L\{(K*u)(t)\} = L\{K(t)\}U(s) = G(s)U(s) \]
Transfer function and stability

- From our knowledge of time domain solutions
  - If poles are distinct system is
    - Asymptotically stable if and only if $\text{Re}[p_i] < 0, \forall i$
    - Stable if and only if $\forall i \text{ Re}[p_i] \leq 0$
    - Unstable if and only if $\exists i : \text{Re}[p_i] > 0$
  - If poles are repeated system is
    - Asymptotically stable if and only if $\text{Re}[p_i] < 0, \forall i$
    - Unstable if $\exists i : \text{Re}[p_i] > 0$
    - If $\forall i \text{ Re}[p_i] \leq 0$ and $\exists i \text{ Re}[p_i] = 0$ system may be stable or unstable, depending on partial fraction expansion
      (cf. “depending on eigenvectors” of matrix $A$, Notes 3)
    - Provided there are no pole zero cancellations
Block diagrams

\[ G_1(s) \rightarrow G_2(s) \Leftrightarrow G_2(s)G_1(s) \]

\[ G_1(s) \rightarrow G_2(s) \rightarrow \oplus \rightarrow G_2(s)G_1(s) \Leftrightarrow G_2(s)+G_1(s) \]

\[ G_1(s) \rightarrow G_2(s) \rightarrow \oplus \rightarrow \oplus \rightarrow [1+G_1(s)G_2(s)]^{-1}G_1(s) \]

Caution: MIMO transfer functions in general matrices!

5.20
$Y(s) = [1 + G(s)K_2(s)K_3(s)]^{-1}G(s)K_2(s)K_1(s)U(s)$

- In the SISO case: Composition or rational transfer functions is also a rational transfer function
- Properties of “closed loop” system studied using the same tools

**Exercise:** In the SISO case, show that if $G(s)$ strictly proper, $K_1(s)$, $K_2(s)$, $K_3(s)$ proper then closed loop transfer function is strictly proper
Frequency response

- In RLC example, steady state response to sinusoidal input is sinusoidal
- More generally consider proper, stable SISO system with transfer function $G(s)$
- Apply $u(t) = \sin(\omega t)$
- Output settles to sinusoid $y(t) = K \sin(\omega t + \phi)$ with
  - The same frequency, $\omega$
  - Amplitude $K = |G(j\omega)| = \sqrt{\text{Re}[G(j\omega)]^2 + \text{Im}[G(j\omega)]^2}$
  - Phase $\phi = \angle G(j\omega) = \tan^{-1}\left(\frac{\text{Im}[G(j\omega)]}{\text{Re}[G(j\omega)]}\right)$
- Shown by partial fraction expansion of $Y(s) = G(s) \frac{\omega}{s^2 + \omega^2}$
Frequency response

- Response of system to sinusoids at different frequencies called the frequency response
- Frequency response important because
  - Sinusoids are common inputs
  - Directly related to any other input by Fourier transform
  - Frequency response tells us a lot about system behavior
  - E.g. Will it be stable under various interconnections?
- Frequency response usually summarized graphically
  - Bode plots: Log-log plot $|G(j\omega)|$ vs. $\omega$, lin-log plot $\angle G(j\omega)$ vs. $\omega$
  - Nyquist plot: $G(j\omega)$ in polar coordinates, parameterized by $\omega$
  - Nichols chart: Log-lin plot $|G(j\omega)|$ vs. $\angle G(j\omega)$, parameterized by $\omega$
Bode plots (bode.m)

Pair of plots, x-axis the same \( \log(\omega) \) (in rad/sec), y-axes \( 20\log(\|G(j\omega)\|) \) (in dB) and \( \angle G(j\omega) \) (in degrees)

\[
G(s) = \frac{2}{(s + 1)(s + 2)}
\]

RLC example (p. 5.14)

\[
G(s) = \frac{s}{(s + 1)(s + 2)}
\]
Nyquist plot (nyquist.m)

Plot of $\text{Im}[G(j\omega)]$ vs. $\text{Re}[G(j\omega)]$ parameterized by $\omega$

$$G(s) = \frac{2}{(s + 1)(s + 2)}$$

RLC example (p. 5.14)

$$G(s) = \frac{s}{(s + 1)(s + 2)}$$
Resonance

- Appears in second order systems (two poles)
- Bode magnitude plot has maximum at some frequency
- Sinusoidal inputs around this frequency get amplified
- Important consequences for performance
- Second order systems very common in practice
- Example: Simplified suspension model

\[
M\ddot{x}(t) + d\dot{x}(t) + kx(t) = -f(t)
\]

\[
\frac{X(s)}{F(s)} = -\frac{1}{Ms^2 + ds + k}
\]
Resonance

• Second order transfer functions of interest look like

\[ G(s) = \frac{K \omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}, \quad \omega_n > 0 \]

• For suspension example: \( \omega_n = \sqrt{\frac{k}{M}}, \quad \zeta = \frac{d}{2\sqrt{km}}, \quad K = -\frac{1}{k} \)

• Frequency response

\[ G(j\omega) = \frac{K \omega_n^2}{(\omega_n^2 - \omega^2) + j(2\zeta \omega_n \omega)} \]

\[ |G(j\omega)| = \frac{K \omega_n^2}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta \omega_n \omega\right)^2}} \]

\[ \angle G(j\omega) = -\tan^{-1}\left(\frac{2\zeta \omega_n \omega}{\omega_n^2 - \omega^2}\right) \]
Resonance

1. For stability need $\zeta \geq 0$
2. For $\zeta \geq 1$ poles real (over-damped system)
3. For $\zeta = 1$ poles real and equal (critical damping)
4. For $0 < \zeta < 1$ poles complex (under-damped system)
5. For $\zeta = 0$ poles imaginary (undamped system)

6. For $\zeta \geq \frac{1}{\sqrt{2}}$ magnitude Bode plot decreasing in $\omega$

7. For $0 \leq \zeta < \frac{1}{\sqrt{2}}$ magnitude Bode plot has a maximum

\[ \omega = \omega_n \sqrt{1 - 2\zeta^2} \quad \text{and} \quad |G(j\omega)| = \frac{K}{2\zeta \sqrt{1 - \zeta^2}} \]

Exercise: Verify 1-5
Exercise: Take the derivative of $|G(j\omega)|$ to verify 6-7
Example: AFM Resonances
Transfer function realization

• Time domain description $\rightarrow$ unique transfer function
\[
\begin{align*}
\frac{dx}{dt}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]
\[\rightarrow G(s) = C(sI - A)^{-1}B + D\]

• Transfer function $\rightarrow$ unique time domain description??

\[G(s) = \frac{(s - z_1)(s - z_2)\cdots(s - z_k)}{(s - p_1)(s - p_2)\cdots(s - p_n)} \quad \Rightarrow \quad \begin{cases} 
\frac{dx}{dt}(t) = Ax(t) + Bu(t) \\
y(t) = Cx(t) + Du(t)
\end{cases}\]

• Given $G(s)$, choice of $A$, $B$, $C$, $D$ such that $C(sI - A)^{-1}B + D = G(s)$ known as a realization of $G(s)$

• Clearly not unique, e.g. coordinate change $\hat{x} = Tx$, $\det(T) \neq 0$
Realization: SISO, strictly proper system

- SISO, strictly proper system

\[ G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n} \]

\[ \dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \]

\[ y(t) = \begin{bmatrix} b_n & b_{n-1} & b_{n-2} & \cdots & b_1 \end{bmatrix} x(t) \]

Controllable canonical form

\[ \dot{x}(t) = \begin{bmatrix} 0 & 0 & \cdots & -a_n \\ 1 & 0 & \cdots & -a_{n-1} \\ 0 & 1 & \cdots & -a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -a_1 \end{bmatrix} x(t) + \begin{bmatrix} b_n \\ b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \end{bmatrix} u(t) \]

\[ y(t) = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} x(t) \]

Observable canonical form

Exercise: Show that both the controllable and the observable canonical forms are realizations of \( G(s) \)
Uncontrollable and unobservable systems

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)
\end{align*}
\]

1. In both cases transfer function \( G(s) = \frac{1}{s+1} \)

2. Same as \( \dot{x}(t) = -x(t) + u(t), \ y(t) = x(t) \in \mathbb{R} \)

3. Original state space system unstable

4. Transfer function poles have negative real parts!

5. Pole-zero cancellation of term corresponding to uncontrollable/unobservable part

6. Can be shown in general using Kalman decomposition

Exercise: Show points 1-5
In summary

• Transfer function alternative system description to state space
• Closely related, not equivalent
• Advantages
  + Coordinate independent
  + Easier to manipulate for system composition
  + Easier to compute response to “complicated” inputs
  + Immediate connection to steady state sinusoidal response
  + May also work for systems that do not have state space description (e.g. delay elements)
• Disadvantages
  – Less natural in terms of physical laws
  – Used mostly for ZSR
  – May contain less information than state space description
  – Unobservable and uncontrollable parts lost
Sampled Data Systems

- Computers operate on bit streams
- Value and time quantization
  1. Variables can take finitely many values
  2. Operations performed at fixed "clock" period
Sampled Data Systems

• In “embedded” computational systems digital computer has to interact with analog environment.
  – Measurements of physical quantities processed by computers
  – Decisions of computer applied to physical system

• Requires transformation of real valued signals of real time to discrete valued signals of discrete time and vice-versa
  – Analog to digital conversion (A/D or ADC)
  – Digital to analog conversion (D/A or DAC)
Sampled Data Systems

• Usually value quantization is quite accurate.
• Here we ignore value quantization, we concentrate on time quantization.

• Assume:
  – “ADC” → sample every $T$
  – “DAC” → zero order hold
Sampled Data Systems

\[
\dot{x}(t) = \bar{A}x(t) + \bar{B}u(t)
\]
\[
y(t) = \bar{C}x(t) + \bar{D}u(t)
\]
Sampled Data Linear Systems

How does linear system with sampling and zero order hold look like from computer?

\[
\dot{x}(t) = \bar{A}x(t) + \bar{B}u(t) \quad \bar{A} \in \mathbb{R}^{n \times n} \quad \bar{B} \in \mathbb{R}^{n \times m}
\]
\[
y(t) = \bar{C}x(t) + \bar{D}u(t) \quad \bar{C} \in \mathbb{R}^{p \times n} \quad \bar{D} \in \mathbb{R}^{p \times m}
\]
\[
u(t) = u_k \quad \text{for all } t \in [kT, (k + 1)T)
\]
\[
y_k = y(kT)
\]

For \( t \in [kT, (k + 1)T) \)

\[
x(t) = e^{\bar{A}(t-kT)}x(kT) + \int_{kT}^{t} e^{\bar{A}(t-\tau)}\bar{B}u(\tau)d\tau
\]
Sampled Data Linear Systems

\begin{align*}
x((k+1)T) &= e^{AT}x(kT) + \left( \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)}Bd\tau \right)u_k \\
&= e^{AT}x(kT) + \left( \int_{0}^{T} e^{A(T-\tau)}Bd\tau \right)u_k \\
y(kT) &= \bar{C}x(kT) + \bar{D}u(kT)
\end{align*}

**Exercise:** Show that
\[ \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)}Bd\tau = \int_{0}^{T} e^{A(T-\tau)}Bd\tau \]

Let \( x_k = x(kT) \), \( u_k = u(kT) \), \( y_k = y(kT) \). Then
\[ x_{k+1} = Ax_k + Bu_k \]
\[ y_k = Cx_k + Du_k \]

with \( A = e^{AT} \), \( B = \int_{0}^{T} e^{A(T-\tau)}Bd\tau \),
\[ C = \bar{C}, \quad D = \bar{D} \]
Discrete Time Linear Systems

\[ x_{k+1} = Ax_k + Bu_k \]
\[ y_k = Cx_k + Du_k \]

Solution: Given \( \hat{x}_0 \in \mathbb{R}^n \) and \( u_k \in \mathbb{R}^m, k = 0,1,\ldots, N - 1 \) solution consists of two sequences \( x_k \in \mathbb{R}^n, k = 0,1,\ldots, N \) and \( y_k \in \mathbb{R}^p, k = 0,1,\ldots, N \) such that:

\[ x_0 = \hat{x}_0 \]
\[ x_{k+1} = Ax_k + Bu_k, \quad k = 0,1,\ldots, N - 1 \]
\[ y_k = Cx_k + Du_k, \quad k = 0,1,\ldots, N \]

\[ x_k \in \mathbb{R}^n, u_k \in \mathbb{R}^m, y \in \mathbb{R}^p \]
\[ A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m} \]
\[ C \in \mathbb{R}^{p \times n}, D \in \mathbb{R}^{p \times m} \]
Solution of Discrete Time Linear Systems

\[ x_k = A^k \hat{x}_0 + \sum_{i=0}^{k-1} A^{k-i-1} Bu_i \]

ZIT

ZST

Exercise: Prove this by induction

ZST: Discrete time convolution of
- Input \( u_i \)
- State impulse response \( h_k = A^{k-1} B \)

\( y_k \) can easily be computed by:
\[ y_k = C x_k + D u_k \]
Computation of solution

- Hard part is computation of $A^k$ (c.f. $e^{At}$)
- If matrix is diagonalizable

$$A = W \Lambda W^{-1} \Rightarrow A^k = W \Lambda^k W^{-1}$$

$$\Lambda^k = \begin{bmatrix}
\lambda_1^k & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & \lambda_n^k 
\end{bmatrix}$$

**Definition:** The system is called **stable** if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x_0\| \leq \delta \Rightarrow \|x_k\| \leq \varepsilon$ for all $k = 0, 1, \ldots$. It is called **asymptotically stable** if in addition $\lim_{k \to \infty} \|x_k\| = 0$. A system that is not stable is called **unstable**.
Stability, diagonalizable matrices

- If matrix diagonalizable, $A^k$ linear combination of $\lambda_i^k$
  
  - $\lambda_i = \sigma_i \pm j\omega_i, |\lambda_i| = \sqrt{\sigma_i^2 + \omega_i^2}$
  
  - $|\lambda_i| < 1 \Rightarrow |\lambda_i|^k \xrightarrow{k \to \infty} 0$
  
  - $|\lambda_i| = 1 \Rightarrow |\lambda_i|^k = 1 \forall k$
  
  - $|\lambda_i| > 1 \Rightarrow |\lambda_i|^k \xrightarrow{k \to \infty} \infty$

**Exercise:** Show that if $\overline{A}$ is diagonalizable and $\forall i, \text{Re}[\overline{\lambda}_i] < 0$ then $A = e^{\overline{A}T}$ is diagonalizable and $\forall i, |\overline{\lambda}_i| < 1$.

**Theorem 6.1:** System with diagonalizable $A$ matrix is:

- Stable if and only if $\forall i |\lambda_i| \leq 1$
- Asymptotically stable if and only if $\forall i |\lambda_i| < 1$
- Unstable if and only if $\exists i : |\lambda_i| > 1$
As before, eigenvalues not enough to determine stability

Case $\forall i \left| \lambda_i \right| \leq 1$ may be either stable, or unstable depending on repetition pattern of eigenvalues with $\left| \lambda_i \right| = 1$ (determined by eigenvectors)

The analogy to continuous time systems is not always perfect however!
Deadbeat response

• Assume all eigenvalues of $A$ are zero:
  \[ \lambda_1 = \lambda_2 = \ldots = \lambda_n = 0 \]

• Then $A^N = 0$ for some $N \leq n$ (nilpotent matrix)

• Example:
  \[ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow A^3 = 0 \]

  \[ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow A^2 = 0 \]

  \[ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow A^N = 0 \]

• Then $x_k = A^k x_0 = 0$ for all $k \geq N$

• ZIT gets to 0 in finite time and stays there.

• This never happens with continuous time systems.
Coordinate change

- Assume $\hat{x}_k = Tx_k$ for some invertible $T \in \mathbb{R}^{n \times n}$
- In the new coordinates system dynamics are again linear time invariant

$$
\hat{x}_{k+1} = \hat{A}\hat{x}_k + \hat{B}u_k \\
y_k = \hat{C}\hat{x}_k + \hat{D}u_k
$$

Exercise: Show this

with

$$
\hat{A} = TAT^{-1}, \quad \hat{B} = TB \\
\hat{C} = CT^{-1}, \quad \hat{D} = D
$$
Energy and Power

• Consider “energy like” function:
  \[ V(x) = \frac{1}{2} x^T Q x \quad Q = Q^T > 0 \]

• “Power”: change of energy in time
  \[
  V(x_{k+1}) = \frac{1}{2} x_{k+1}^T Q x_{k+1} \\
  = \frac{1}{2} x_k^T (A^T QA) x_k + \frac{1}{2} u_k^T B^T Q Bu_k + \\
  + \frac{1}{2} u_k^T B^T Q A x_k + \frac{1}{2} x_k^T A^T Q B u_k
  \]

• If \( u_k = 0 \) (autonomous system)
  \[
  V(x_{k+1}) - V(x_k) = \frac{1}{2} x_k^T (A^T QA - Q) x_k = -\frac{1}{2} x_k^T Rx_k
  \]
  \[
  R = -(A^T QA - Q)
  \]
Stability and Energy

• If $R = R^T > 0$ then energy decreases all the time

• Natural to assume that system is stable

Theorem 6.3: $|\lambda_i| < 1$ for all $i=1, 2, \ldots, n$ if and only if for all $R = R^T > 0$ the equation $(A^TQ - Q) = -R$ has a unique solution with $Q = Q^T > 0$.

Exercise: Show that $R = R^T$
Controllability

- System is **controllable** if we can steer it from any initial condition $\hat{x}_0 \in \mathbb{R}^n$ to any final condition $\hat{x}_N \in \mathbb{R}^n$ using appropriate sequence $u_k, k = 0, 1, \ldots, N - 1$
- Assume $N \geq n$
- Define again controllability matrix

\[
P = [B \quad AB \quad A^2B \cdots A^{n-1}B] \in \mathbb{R}^{n\times nm}
\]

**Theorem 6.4:** The system is controllable if and only if $P$ has rank $n$.

**Exercise:** Prove this
Observability

• System is **observable** if we can infer the state evolution $x_k, k = 0, 1, \ldots, N$ by observing the input and output sequences $u_k, y_k, k = 0, 1, \ldots, N$

• Assume $N \geq n - 1$

• Define again observability matrix

$$Q = \begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{bmatrix} \in \mathbb{R}^{np \times n}$$

**Theorem 6.5:** The system is observable if and only if $Q$ has rank $n$.

**Exercise:** Prove this
z Transform

- Time function $f_k$ converts to a complex variable function $F(z)$
  
  
  $f : \mathbb{N} \rightarrow \mathbb{R}$ \quad $F : \mathbb{C} \rightarrow \mathbb{C}$
  
  \[
  f_k \xrightarrow{Z} F(z) \quad F(z) = Z\{f_k\} = \sum_{k=0}^{\infty} f_k z^{-k}
  \]

- We implicitly assume that $f_k = 0$ for all $k < 0$ (cf. p.0.22)

- Can also be defined for matrix valued functions by taking sum element by element

- $z \in \mathbb{C}$ can be thought of as unit time delay
### $z$ Transform: Properties

**Assumption:** The function $f_k$ is such that the sum converges

- **Linearity** $\mathcal{Z}\left\{a_1 f_k + a_2 g_k\right\} = a_1 F(z) + a_2 G(z)$
- **Time shift** $\mathcal{Z}\left\{f_{k-k_0}\right\} = z^{-k_0} F(z)$
- **Convolution** $\mathcal{Z}\left\{(f * g)_k\right\} = \mathcal{Z}\left\{\sum_{i=0}^{k} f_i g_{k-i}\right\} = F(z)G(z)$

**Exercise:** Prove these

- **Some common functions:**
  - Impulse function $\mathcal{Z}\left\{\delta_k\right\} = 1 \quad (\delta_0 = 1, \delta_k = 0 \text{ if } k \neq 0)$
  - Step function $\mathcal{Z}\left\{1_k\right\} = \frac{z}{z-1} \quad (1_k = 1 \text{ if } k \geq 0, 1_k = 0 \text{ if } k < 0)$
  - Geometric progression $\mathcal{Z}\left\{a^k\right\} = \frac{z}{z-a} \quad (|a| < 1)$
Transfer function

- Assume $x_0 = 0$
- Take $z$ transform of all signals

\[
x_{k+1} = Ax_k + Bu_k \Rightarrow zX(z) = AX(z) + BU(z)
\]

\[
y_k = Cx_k + Du_k \Rightarrow Y(z) = CX(z) + DU(z)
\]

\[
Y(z) = \left[ C \left( zI - A \right)^{-1} B + D \right] U(z)
\]

Exercise: Show that the transfer function is $z$-transform of “impulse response” (appropriately defined!)
Transfer function

\[ G(z) = C(zI - A)^{-1} B + D \]

- Rational function of \( z \).
- System asymptotically stable \( \Leftrightarrow \) Poles of \( G(z) \) have magnitude less than 1

- If system uncontrollable/unobservable pole zero cancellations.
Simulation

- Simulation: Numerical solution in computer
- Simulation of discrete time systems (linear or non-linear) is very easy conceptually
- Discrete time systems can also help understand the simulation of continuous time systems
- Consider continuous time, LTI system

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

- Given \( \hat{x}_0 \in \mathbb{R}^n, u(\cdot) : [0,T] \rightarrow \mathbb{R}^m \) solution is \( x(\cdot) : [0,T] \rightarrow \mathbb{R}^n \), with

\[ x(t) = e^{At}(t)\hat{x}_0 + \int_0^t e^{A(t-\tau)}Bu(\tau) \, d\tau \]
Example: RLC circuit (p. 3.18)

\[ \frac{d}{dt} \begin{bmatrix} v_c(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} v_c(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v_s(t) \]

- Solution depends on eigenvalues and eigenvectors
- Determined by \( R, L, C \)
- Consider autonomous case \( v_s(t) = 0 \) for all \( t \)
For example

\[ R = 3 \, W, \, L = 1 \, H, \, C = 0.5 \, F \]
\[ \lambda_2 = -2 < -1 = \lambda_1 < 0 \]

\[ R = 3 \, W, \, L = 1 \, H, \, C = 0.005 \, F \]
\[ \lambda_i = -1.5 \pm 14.06 \, j \]
For example

\[ R = 0 \Omega, \; L = 1 \text{H}, \; C = 0.005 \text{F} \]

\[ \lambda_i = \pm 14.14 j \]
Numerical approximation

• Approximate the solution with a sequence \( \{x_k\}_{k=0}^{N} \)
• Divide the interval \([0, T]\) in \(N\) equal subintervals
• Let \( \delta = \frac{T}{N} \)
• We approximate

\[
x((k+1)\delta) \approx x(k\delta) + \delta \dot{x}(k\delta) = x(k\delta) + \delta(Ax(k\delta) + Bu(k\delta))
\]

• So we set \( x_0 = x(0) = \hat{x}_0, x_k = x(k\delta), u_k = u(k\delta) \)

\[
x_{k+1} = (I + A\delta)x_k + \delta Bu_k
\]
Numerical approximation

Integration using Euler method

First order approximation of the equation

\[ x(t) = \Phi(t)x_0 + \int_0^t \Phi(t - \tau)Bu(\tau)d\tau \]
Zero input response

- Consider autonomous system
- Solution is $x(t) = \Phi(t)x_0 = e^{At}x_0$

\[
x((k + 1)\delta) = e^{A((k+1)\delta-k\delta)}x(k\delta) = e^{A\delta}x(k\delta)
= \left(I + A\delta + \frac{A^2\delta^2}{2} + \ldots\right)x(k\delta)
\]

- First order approximation

\[
x((k + 1)\delta) \approx (I + A\delta)x_k
\]

- First order approximation good if $\delta$ “small”
Stability of numerical approximation

- How small should the step be?
- If the eigenvalues of $A$ have negative real part, then the system asymptotically stable and $x(t) \to 0$
- At the very least we would like to guarantee that the numerical approximation is such that $x_k \to 0$
- Assume $A$ diagonalizable

\[ A = W \Lambda W^{-1} \]

- Eigenvalue matrix (diagonal)
- Eigenvector matrix (invertible)
Stability of numerical approximation

- Then \( x_k = W (I + \lambda \delta)^k W^{-1} x_0 \)

\[
x_k = W \begin{bmatrix} (1 + \delta \lambda_1)^k & 0 & \ldots & 0 \\ 0 & (1 + \delta \lambda_2)^k & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & (1 + \delta \lambda_n)^k \end{bmatrix} W^{-1} x_0
\]

- Discrete time system asymptotically stable if and only if \( x_k \to 0 \ \forall x_0 \in \mathbb{R}^n \iff 1 + \lambda_i \delta < 1 \ \forall i = 1, \ldots, n \)

- For example, if \( \lambda_i \) are real and negative

\[
\delta < \frac{2}{\max_{i=1, \ldots, n} |\lambda_i|}
\]

Exercise: Prove this

Exercise: Repeat for complex eigenvalues
RLC circuit with $R=3\,\Omega$, $L=1\,H$, $C=0.5\,F$

$\delta = 0.01$

$\delta = 0.05$
RLC circuit with $R=3\,\Omega$, $L=1\,H$, $C=0.5\,F$

«Exact» solution

Numerical approximation

Instability!

$\delta = 0.25$

$\lambda_1 = -1$, $\lambda_2 = -2 \Rightarrow \delta < 1$ for stability
Simulation

- Simple first order approximation known as “forward Euler” method
- Another approach is “backward Euler”

\[ \dot{x} = Ax \Rightarrow x_{k+1} \approx x_k + \delta A x_{k+1} \Rightarrow x_{k+1} \approx (I - \delta A)^{-1} x_k \]

- Care also needed when selecting step \( \delta \)
- Much better methods than Euler exist
  - E.g. Runge-Kutta, variable step, high order
  - Specialized methods for “stiff” systems, hybrid systems, differential-algebraic systems, etc.
  - Coded in robust numerical tools such as Matlab
Signal- und Systemtheorie II
D-ITET, Semester 4

Notes 7: Nonlinear systems

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Nonlinear systems

• Most of this course: Dynamical systems modeled by linear differential equations in state space form

\[ \dot{x}(t) = Ax(t) + Bu(t) \quad x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p \]

\[ y(t) = Cx(t) + Du(t) \quad A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, D \in \mathbb{R}^{p \times m} \]

• Last few lectures return to more general systems

\[ \dot{x}(t) = f(x(t), u(t)) \quad x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p \]

\[ y(t) = h(x(t), u(t)) \quad f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n, h : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^p \]

• Concentrate on continuous time

• Discrete time can be more complicated
  – E.g. the “Population Dynamics” example in Notes 1 is one dimensional but can be chaotic
Nonlinear systems

- More general than linear system, hence more difficult
- Concentrate on autonomous, time invariant systems

\[ \dot{x}(t) = f(x(t)) \quad \left( \text{In the linear case } \dot{x}(t) = Ax(t) \right) \]

- Assume function \( f \) is Lipschitz

\[ \exists \lambda > 0, \forall x, \hat{x} \in \mathbb{R}^n, \quad \| f(x) - f(\hat{x}) \| \leq \lambda \| x - \hat{x} \| \]

- This implies existence and uniqueness of solutions
- In general solution cannot be computed analytically
- Simulation methods applicable however
- Look into the following issues
  - Invariant sets
  - Stability of invariant sets
Invariant sets

• Generalization of notion of equilibrium

**Definition:** A set of states \( S \subseteq \mathbb{R}^n \) is called **invariant** if

\[
\forall x_0 \in S, \forall t \geq 0, \quad x(t) \in S
\]

\( x(t) \) means the solution to \( \dot{x}(t) = f(x(t)) \) starting at \( x_0 \)

• Equilibrium points are an important class of invariant sets

**Definition:** A state \( \hat{x} \in \mathbb{R}^n \) is called an **equilibrium** if

\[
f(\hat{x}) = 0
\]

**Exercise:** Prove that if \( \hat{x} \) is an equilibrium then \( S = \{ \hat{x} \} \) is an invariant set
Equilibria

- Linear systems have a linear subspace of equilibria
  - Sometimes only $\hat{x} = 0$
  - More generally, the null space of the matrix $A$

**Exercise:** Show that the equilibria of $\dot{x}(t) = Ax(t)$ coincide with the null space of $A$

- Nonlinear systems can have many isolated equilibria
- Example: The pendulum from Notes 1 has 2 equilibria

\[ \dot{x}(t) = \begin{bmatrix} x_2(t) \\ -\frac{d}{m} x_2(t) - \frac{g}{l} \sin x_1(t) \end{bmatrix} \Rightarrow \hat{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \hat{x}' = \begin{bmatrix} \pi \\ 0 \end{bmatrix} \]

(More precisely, number of pendulum equilibria is infinite, but they all coincide physically with these two)
Exercise: Let $x_{k+1} = f(x_k)$ be a nonlinear system in discrete time (cf. p.1.27). The equilibria for this system are given by

$$\hat{x} = f(\hat{x})$$

Show that equilibria are invariant sets (cf. p.1.29).
Shifting equilibria to the origin

• It is often convenient to “shift” an equilibrium to the origin before analyzing the system behavior.

• This involves a change of coordinates

\[ w(t) = x(t) - \hat{x} \in \mathbb{R}^n \]

• In the new coordinates the system becomes

\[ \dot{w}(t) = \dot{x}(t) = f(x(t)) = f(w(t) + \hat{x}) = \hat{f}(w(t)) \]

• The system in the new coordinates has an equilibrium at \( \hat{w} = 0 \in \mathbb{R}^n \)

Exercise: Show this
Limit cycles

- Observed only in systems of dimension 2 or more

**Definition:** A solution $x(t)$ is called a **periodic orbit** if

$$\exists T > 0, \forall t \geq 0, \quad x(t + T) = x(t)$$

- Equilibria define trivial periodic orbits
- Limit cycles: Non-trivial periodic orbits
- Linear systems exhibit either
  - Trivial periodic orbits (equilibria)
  - Subspaces of periodic orbits, e.g.

$$\dot{x}(t) = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x(t)$$

- Nonlinear systems can also have non-trivial, isolated periodic orbits $\Rightarrow$ **Limit cycles**

**Exercise:** Show that solution starting at an equilibrium is periodic

**Exercise:** Show this system has subspace of periodic orbits
Example: Van der Pol oscillator

- Developed as a model for dynamics of vacuum tube (transistor) circuits
- Under certain conditions circuits observed to oscillate
- Van der Pol showed this is due to “nonlinear resistance” phenomena
- Second order differential equation

\[ \ddot{\theta}(t) - \varepsilon (1 - \theta(t)^2) \dot{\theta}(t) + \theta(t) = 0 \]

**Exercise:** Write the equation for the van der Pol oscillator in state space form. Hence determine its equilibria.
Example: van der Pol oscillator, $\varepsilon=1$

Exercise: Let $x_{k+1} = f(x_k)$ be a nonlinear system in discrete time. How would you define periodic orbits and limit cycles for this system? (cf. p.1.28).

Stable limit cycle

Unstable equilibrium
Strange attractors

- In 2D continuous time equilibria & limit cycles as bad as it gets (Poincare-Bendixson Theorem)
- In higher dimensions stranger things may happen
  - Invariant tori
  - Chaotic attractors
- Example: Lorenz equations
  - Developed by E.N. Lorenz
  - To capture atmospheric phenomena

\[
\begin{align*}
\dot{x}_1(t) &= a(x_2(t) - x_1(t)) \\
\dot{x}_2(t) &= (1 + b)x_1(t) - x_2(t) - x_1(t)x_3(t) \\
\dot{x}_3(t) &= x_1(t)x_2(t) - cx_3(t)
\end{align*}
\]
Chaotic attractor

• For some parameter values, there is a bounded subset of the state space such that if we start inside we stay there for ever and
  – Most trajectories go around for ever,
  – Without ever meeting themselves (not limit cycles)
• Given any two points in this set we can find a trajectory that starts arbitrarily close to one and ends up arbitrarily close to the other
• This set is called a chaotic or strange attractor

Exercise: Compute the equilibria of the Lorenz equations

Exercise: Simulate the Lorenz equations for $a=10$, $b=24$, $c=2$, and $x_0=(-5, -6, 20)$
Lorenz attractor simulation
Stability

- Most commonly studied property of invariant sets
- Trajectories stay close or converge to invariant set
- Restrict attention to equilibria
- Simple characterization for LTI and equilibrium $\hat{x} = 0$
  - Systems stable if eigenvalues of $A$ have negative real part
  - Poles of transfer function are in left half of complex plane

**Definition:** An equilibrium $\hat{x}$ is called **stable** if for all $\varepsilon > 0$ there exists $\delta > 0$ such that
\[
\left\| x_0 - \hat{x} \right\| < \delta \implies \left\| x(t) - \hat{x} \right\| < \varepsilon \quad \forall t \geq 0
\]
Otherwise equilibrium called **unstable**.

**Exercise:** Which of the equilibria of the pendulum (simulation p. 7.6) would you say are stable and which not?
Asymptotic stability

- Stability says that if we start close we stay close
- Do we get closer and closer?

**Definition**: An equilibrium \( \hat{x} \) is called **locally asymptotically stable** if it is stable and there exists \( M > 0 \) such that

\[
\| x_0 - \hat{x} \| < M \Rightarrow \lim_{t \to \infty} x(t) = \hat{x}
\]

It is called **globally asymptotically stable** if this holds for any \( M > 0 \).

The set of \( x_0 \) such that \( \lim_{t \to \infty} x(t) = \hat{x} \) is called the **domain of attraction** of \( \hat{x} \).

**Exercise**: What is the domain of attraction of a globally asymptotically stable equilibrium?

**Exercise**: Is there a difference between local and global asymptotic stability for linear systems?
Example: Pendulum with $d > 0$

Exercise: Which of the equilibria would you say are locally asymptotically stable? Which globally?
Linearization

• Simple way to study stability of equilibrium of nonlinear system is to approximate by linear system

\[ \dot{x}(t) = f(x(t)), \quad f(\hat{x}) = 0 \]

• Take Taylor expansion about \( \hat{x} \)

\[ f(x) = f(\hat{x}) + A(x - \hat{x}) + \text{higher order terms in } (x - \hat{x}) \]

\[ = A(x - \hat{x}) + \text{higher order terms in } (x - \hat{x}) \]

\[
x = \begin{bmatrix}
x_1 \\
\vdots \\
x_n 
\end{bmatrix},
\quad f(x) = \begin{bmatrix}
f_1(x_1, \ldots, x_n) \\
\vdots \\
f_n(x_1, \ldots, x_n)
\end{bmatrix},
\quad A = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1}(\hat{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\hat{x}) \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1}(\hat{x}) & \cdots & \frac{\partial f_n}{\partial x_n}(\hat{x})
\end{bmatrix} \in \mathbb{R}^{n \times n}
\]
Linearization

- Consider distance of $x$ to equilibrium $\delta x(t) = x(t) - \hat{x} \in \mathbb{R}^n$
- When $x$ close to equilibrium, $\delta x$ is small and

$$\frac{d\delta x(t)}{dt} \approx A\delta x(t)$$

- So close to equilibrium nonlinear system expected to behave like a linear system
- In particular, stability of the linearization should tell us something about stability of nonlinear system
- Stability of linearization can be determined just by looking at the eigenvalues of $A$
Stability by linearization

**Theorem 7.1:** The equilibrium \( \hat{x} \) is
1. Locally asymptotically stable if the eigenvalues of the linearization have negative real part
2. Unstable if the linearization has at least one eigenvalue with positive real part

- Called **Lyapunov first** or **Lyapunov indirect method**
- Advantage: Very easy to use
- Disadvantages:
  - No information about the domain of attraction
  - Inconclusive if linearization has imaginary/zero eigenvalues
Pendulum example, $d>0$

- Linearization about $\hat{x} = (0,0)$

$$\frac{d\delta x(t)}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{d}{m} \end{bmatrix} \delta x(t) \Rightarrow \lambda^2 + \frac{d}{m} \lambda - \frac{g}{l} = 0$$

- Eigenvalues have negative real part, hence equilibrium locally asymptotically stable

- Linearization about $\hat{x} = (\pi,0)$

$$\frac{d\delta x(t)}{dt} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{d}{m} \end{bmatrix} \delta x(t) \Rightarrow \lambda^2 + \frac{d}{m} \lambda - \frac{g}{l} = 0$$

- At least one eigenvalue has positive real part, hence equilibrium unstable
Linearization can be inconclusive

- Notice that if $d=0$
  - Linearization about $\hat{x}=(\pi,0)$ has positive eigenvalue
  - Hence $\hat{x}=(\pi,0)$ is unstable for nonlinear system
  - Linearization about $\hat{x}=(0,0)$ has imaginary eigenvalues
  - Stability of $\hat{x}=(0,0)$ not determined from Theorem 7.1

- It turns out that equilibrium is stable (see fig. on p.7.6)
- This is not always the case
- For example, the linearization of both

$$\dot{x}(t) = x(t)^3 \quad \text{and} \quad \dot{x}(t) = -x(t)^3$$

about $\hat{x} = 0$ has one eigenvalue at zero
- But $0$ stable for one system and unstable for the other
Lyapunov functions

• In linear systems stability characterized in two ways
  – Eigenvalues of matrix $A$ (Theorems 3.1, 3.2), or poles of the transfer function (p.5.19)
  – Existence of decreasing energy-like function (Theorem 4.1)
• First applies to nonlinear systems, how about second?
• Properties of energy-like function for linear systems
  1. Quadratic function of the state $V(x) = \frac{1}{2} x^T Q x$
  2. $Q$ positive definite $\Rightarrow V(x)>0$ for all $x \neq 0$, $V(0) = 0$
  3. Power also quadratic of the state $\frac{d}{dt} V(x) = -\frac{1}{2} x^T R x$
  4. $R = -(A^T Q + QA)$ positive definite $\Rightarrow V(x)$ decreases for all $x \neq 0$
• For nonlinear systems keep 2 and 4, but allow more general (non-quadratic) $V(x)$
Lyapunov functions: Stability

Theorem 7.2: Assume there exists an open set $S \subset \mathbb{R}^n$ with $\hat{x} \in S$ and a differentiable function $V(\cdot): \mathbb{R}^n \rightarrow \mathbb{R}$

1. $V(\hat{x}) = 0$
2. $V(x) > 0$, $\forall x \in S$ with $x \neq \hat{x}$
3. $\frac{d}{dt} V(x(t)) \leq 0$, $\forall x \in S$

Then the equilibrium $\hat{x}$ is stable

- Called **Lyapunov second** or **Lyapunov direct method**
- Function $V(x)$ known as **Lyapunov function**
- Derivative along trajectories known as **Lie derivative**

$$\frac{d}{dt} V(x(t)) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(x(t)) \frac{d}{dt} x_i(t) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(x(t)) f_i(x(t)) = \nabla V(x(t)) f(x(t))$$
"Proof": By picture!

$$S_c = \left\{ x \in S \mid V(x) \leq c \right\}$$
Example: Pendulum for $d=0$

- Recall that linearization could not determine the stability of $\hat{x} = (0, 0)$ when $d=0$
- Consider the energy

\[
V(x) = \frac{1}{2} m (l \dot{\theta})^2 + mgl(1 - \cos(\theta))
\]

\[
= \frac{1}{2} ml^2 x_2^2 + mgl(1 - \cos(x_1))
\]
Example: Pendulum for $d=0$

Take $S = (-\pi, \pi) \times \mathbb{R}$ check theorem conditions

1. $V(0) = 0$
2. $V(x) > 0 \quad \forall x \neq 0$
3. $\frac{d}{dt}V(x(t)) = ml^2 x_2(t) \dot{x}_2(t) + mgl \sin(x_1(t))\dot{x}_1(t) = 0$

Hence the equilibrium is stable
Lyapunov functions: Asymptotic stability

**Theorem 7.3**: Assume there exists an open set \( S \subseteq \mathbb{R}^n \) with \( \hat{x} \in S \) and a differentiable function \( V(\bullet): \mathbb{R}^n \rightarrow \mathbb{R} \)

1. \( V(\hat{x}) = 0 \)
2. \( V(x) > 0, \forall x \in S \) with \( x \neq \hat{x} \)
3. \( \frac{d}{dt} V(x(t)) < 0, \forall x \in S \) with \( x \neq \hat{x} \)

Then the equilibrium \( \hat{x} \) is locally asymptotically stable. If \( S = \mathbb{R}^n \) then it is globally asymptotically stable.

- Lyapunov functions can help estimate domain of attraction. If we can find \( c > 0 \) such that
  \[
  \left\{ x \in \mathbb{R}^n \mid V(x) \leq c \right\} \subseteq S
  \]
  then trajectories that start in this set stay in it and converge to \( \hat{x} \)
Examples

• Consider first
  \[ \dot{x}(t) = f(x(t)) = -x(t)^3 \quad \text{where } \dot{x} = 0 \]

• Let \( S = \mathbb{R}, \quad V(x) = x^2 \)

• Clearly \( V(0) = 0, V(x) > 0 \ \forall x \neq 0, \frac{\partial}{\partial x} V(x)f(x) = -2x^4 < 0 \ \forall x \neq 0 \)

• Therefore 0 is globally asymptotically stable

• How about pendulum with \( d > 0 \)

• As before consider \( S = (-\pi, \pi) \times \mathbb{R} \) and \( V(x) \) the energy

  \[
  \frac{d}{dt} V(x(t)) = ml^2 \dot{x}_2(t) \ddot{x}_2(t) + mgl \sin(x_1(t)) \dot{x}_1(t) = -dl^2 x_2(t)^2 \leq 0
  \]

• But =0 whenever \( x_2(t) = 0 \) (not only at \( \dot{x} = (0,0) \)), therefore cannot conclude local asymptotic stability
La Salle’s Theorem

**Theorem 7.4:** Assume there exists a compact invariant set $S \subseteq \mathbb{R}^n$ and a differentiable function $V(\cdot): \mathbb{R}^n \to \mathbb{R}$ such that

$$\nabla V(x)f(x) \leq 0 \quad \forall x \in S$$

Let $M$ be the largest invariant set contained in the set

$$\bar{S} = \left\{ x \in S \mid \nabla V(x)f(x) = 0 \right\} \subseteq \mathbb{R}^n$$

Then all trajectories starting in $S$ tend to $M$ as $t \to \infty$.

- “Compact” means bounded and closed
- If $\hat{x}$ only invariant set in

$$\left\{ x \in S \mid \nabla V(x)f(x) = 0 \right\}$$

then all trajectories starting in $S$ tend to it.
Pendulum with $d > 0$

- Take $V(x)$ the energy and $S = \left\{ x \in \mathbb{R}^2 \mid V(x) \leq 2mgl - \varepsilon \right\}$ for any $\varepsilon > 0$

Exercise: Show that $S$ is invariant

- Recall that

$$\nabla V(x) f(x) = -dl^2x_2^2 \begin{cases} 
\leq 0 & \forall x \in S \\
= 0 & \text{when } x_2 = 0
\end{cases}$$

Energy when pendulum stopped upside down
Pendulum with $d > 0$

- Therefore
  \[\bar{S} = \{ x \in S | x_2 = 0 \}\]
- $\hat{x} = (0,0)$ is the only invariant set contained in $\bar{S}$, since $\dot{x}_2 \neq 0$ if $x_2 = 0$ but $x_1 \neq 0$
- Therefore all trajectories that start in $S$ tend to $\hat{x} = (0,0)$
- By Theorem 7.2, $\hat{x} = (0,0)$ is stable
- Hence, by Theorem 7.4, locally asymptotically stable
- Moreover, since $\epsilon$ is arbitrary, the domain of attraction of $(0,0)$ contains everything except the other equilibrium $(\pi,0)$
General comments

- Theorem 7.4 applies to more general invariant sets (e.g. limit cycles)
- Theorems 7.2 and 7.3 also generalize easily
- Theorem 7.1 slightly harder to generalize (linearization about trajectories, Poincare maps)
- Conditions of Theorems 7.2-7.4 sufficient and not necessary
- Finding Lyapunov functions for nonlinear systems an art not a science. Common choices
  - Energy for mechanical and electrical systems
  - Quadratics (always work for linear systems)
  - Intuition!