Decision making under uncertainty: Performance bounds for the scenario approach[1]

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[1] "Performance bounds for the scenario approach and an extension to a class of non-convex programs", *IEEE Transactions on Automatic Control*, vol. 60, no. 1, 2015

Random convex programs

Robust program (RP)

$$J_{RP} := \begin{cases} \min\limits_x & c^\top x \\ \text{s.t.} & f(x, d) \le 0, \ \ \forall d \in \mathcal{D} \\ & x \in \mathbb{X} \end{cases}$$

- tractable if f and $\mathcal{D}$ are "nice"
- NP-hard in general

Chance constrained program (CCP_ε)

$$J_{\text{CCP}_\varepsilon} := \begin{cases} \min\limits_x & c^\top x \\ \text{s.t.} & \mathbb{P}[f(x, d) \le 0] \ge 1 - \varepsilon \\ & x \in \mathbb{X} \end{cases}$$

- non-convex problem, integration
- NP-hard in general

Scenario program (SP)

$$J_N := \begin{cases} \min\limits_x & c^\top x \\ \text{s.t.} & f(x, d_i) \le 0, \ \ \forall i \le N \\ & x \in \mathbb{X} \\ & \{d_i\}_{i \le N} \text{ i.i.d. on } (\mathcal{D}, \mathbb{P}) \end{cases}$$

- finite convex optimization problem
- optimizer $x_N^\star$

Assumptions

- $x \mapsto f(x, d)$ convex $\forall d \in \mathcal{D}$
- $d \mapsto f(x, d)$ bounded $\forall x \in \mathbb{X}$
- $\mathbb{X} \subset \mathbb{R}^n$ compact, convex

Random convex programs (cont'd)

Robust program (RP)

$$J_{RP} := \begin{cases} \min_x & c^\top x \\ \text{s. t.} & f(x, d) \le 0, \ \ \forall d \in \mathcal{D} \\ & x \in \mathbb{X} \end{cases}$$

Chance constrained program (CCP_ε)

$$J_{\text{CCP}_\varepsilon} := \begin{cases} \min_x & c^\top x \\ \text{s. t.} & \mathbb{P}[f(x, d) \le 0] \ge 1 - \varepsilon \\ & x \in \mathbb{X} \end{cases}$$

Scenario program (SP)

$$(x_N^\star, J_N) : \begin{cases} \min_x & c^\top x \\ \text{s. t.} & f(x, d_i) \le 0, \ \ \forall i \le N \\ & x \in \mathbb{X} \\ & \{d_i\}_{i \le N} \text{ i.i.d. on } (\mathcal{D}, \mathbb{P}) \end{cases}$$

	RP	CCP_ε
SP ($x_N^\star$) feasibility	$f(x_N^\star, d) \overset{?}{\le} 0, \ \ \forall d \in \mathcal{D}$	$\mathbb{P}[f(x_N^\star, d) \le 0] \overset{?}{\ge} 1 - \varepsilon$
SP (J_N) performance	$J_{\text{RP}} - J_N \in [0, ?]$	$J_{\text{CCP}_\varepsilon} - J_N \in [?, ?]$

SP ($x_N^\star$) feasibility

	RP	CCP_ε
SP ($x_N^\star$) feasibility	$f(x_N^\star, d) \overset{?}{\leq} 0, \ \forall d \in \mathcal{D}$	$\mathbb{P}[f(x_N^\star, d) \leq 0] \overset{?}{\geq} 1 - \varepsilon$
SP (J_N) performance	$J_{\text{RP}} - J_N \in [0, ?]$	$J_{\text{CCP}_\varepsilon} - J_N \in [?, ?]$

Example

$$\text{RP} : \begin{cases} \min\limits_x & -x \\ \text{s.t.} & x - d \leq 0, \ \forall d \in [0,1] \\ & x \in [-1, 1] \end{cases} \qquad \text{SP} : \begin{cases} \min\limits_x & -x \\ \text{s.t.} & x - d_i \leq 0, \ \forall i \leq N \\ & x \in [-1, 1] \end{cases}$$

- RP-optimizer $x^\star = 0$
- SP-optimizer $x_N^\star = \min_{i \leq N} d_i$

$$\mathbb{P}^N \left[x_N^\star - d \leq 0, \ \forall d \in [0,1] \right] = 0$$

⇒ no statement possible ☹

SP ($x_N^\star$) feasibility (cont'd)

	RP	CCP_ε
SP ($x_N^\star$) feasibility	☹	$\mathbb{P}[f(x_N^\star, d) \le 0] \overset{?}{\ge} 1 - \varepsilon$
SP (J_N) performance	$J_{\text{RP}} - J_N \in [0, ?]$	$J_{\text{CCP}_\varepsilon} - J_N \in [?, ?]$

Theorem (Campi & Garatti, SIAM 08)

$$\varepsilon, \beta \in (0,1),\ N \ge \tilde{N}(\varepsilon, \beta) := \min\Big\{N \in \mathbb{N} : \sum_{i=1}^{\dim \mathbb{X} - 1} \binom{N}{i} \varepsilon^i (1-\varepsilon)^{N-i} \le \beta\Big\}$$

$$\Rightarrow \quad \mathbb{P}^N[\underbrace{\mathbb{P}[f(x_N^\star, d) \le 0] \ge 1 - \varepsilon}_{x_N^\star \text{ feasible for } \text{CCP}_\varepsilon}] \ge 1 - \beta$$

$$\tilde{N}(\varepsilon, \beta) = \begin{cases} \sim 1/\varepsilon \\ \sim \log 1/\beta \\ \sim \dim \mathbb{X} \\ \sim \#\text{binary} \end{cases}$$

Non-convex setting, e.g. mixed integer problems [Mohajerin Esfahani et al., TAC 2015]

SP ($x_N^\star$) performance

	RP	CCP_ε
SP ($x_N^\star$) feasibility	☹	MC, 08 / PME, 15
SP (J_N) performance	$J_{\mathrm{RP}} - J_N \in [0, ?]$	$J_{\mathrm{CCP}_\varepsilon} - J_N \in [?, ?]$

Theorem (Mohajerin Esfahani et al., TAC 15)

$$\varepsilon, \beta \in (0,1),\ N \geq \tilde{N}(\varepsilon,\beta) := \min\Big\{N \in \mathbb{N} : \sum_{i=1}^{\dim \mathbb{X}-1} \binom{N}{i} \varepsilon^i (1-\varepsilon)^{N-i} \leq \beta\Big\}$$

$$\Rightarrow \qquad \mathbb{P}^N\big[J_{\mathrm{RP}} - J_N \in [0, \ell h(\varepsilon)]\big] \geq 1 - \beta$$

- ℓ: Lipschitz constant of perturbation function of RP
  - a priori estimate via Slater point
- function h, computation depends on $\mathcal{D}$
  - asymptotic convergence $\lim_{\epsilon \to 0} h(\varepsilon) = 0$
  - curse of dimensionality $h(\varepsilon) \sim \varepsilon^{1/\dim \mathcal{D}}$

SP ($x_N^\star$) performance (cont'd)

	RP	CCP_ε
SP ($x_N^\star$) feasibility	☹	MC, 08 / PME, 15
SP (J_N) performance	PME et al., 15	$J_{\mathrm{CCP}_\varepsilon} - J_N \in [?, ?]$

Theorem (Mohajerin Esfahani et al., TAC 15)

$$\varepsilon, \beta \in (0,1),\ N \geq \tilde{N}(\varepsilon, \beta) := \min\left\{ N \in \mathbb{N} : \sum_{i=1}^{\dim \mathbb{X} - 1} \binom{N}{i} \varepsilon^i (1-\varepsilon)^{N-i} \leq \beta \right\}$$

$$\Rightarrow \quad \mathbb{P}^N \left[J_{\mathrm{CCP}_\varepsilon} - J_N \in [-\ell h(\varepsilon), 0] \right] \geq 1 - \beta$$

- ℓ: Lipschitz constant of perturbation function of SP
  - a priori estimate via Slater point
  - a posteriori estimate via dual multipliers of SP
- function h, computation depends on $\mathcal{D}$
- increasing N decreases β (ε is fixed)

Proof (sketch)

(i) Tail probability of the worst-case violation

$$p(x, \delta) := \mathbb{P}\Big[\sup_{v \in \mathcal{D}} f(x, v) - \delta < f(x, d) \Big]$$

(ii) uniform level-set bound $h(\cdot)$ of p if for all $\varepsilon \in [0, 1]$

$$h(\varepsilon) \geq \sup \Big\{ \delta \in \mathbb{R}_+ \;\Big|\; \inf_{x \in \mathbb{X}} p(x, \delta) \leq \varepsilon \Big\}$$

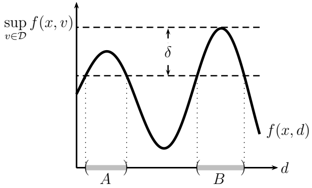

Figure: $p(x, \delta) = \mathbb{P}[A \cup B]$

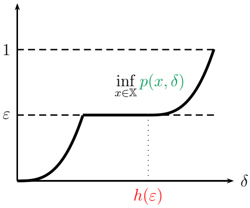

Figure: Uniform level set bound

Applications

Approximate dynamic programming (ADP)

- linear programming approach to ADP (→ robust LP)
- min-max structure → $\ell = 1$
- tail bound function $h(\varepsilon) = L_d c_{\mathcal{D}}\, \varepsilon^{1/\dim \mathcal{D}}$
- [S., Mohajerin Esfahani, and Lygeros, CDC 2015]

Fault detection problem

- robust quadratic program
- [Mohajerin Esfahani and Lygeros, TAC 2015]

Conclusion and Outlook

Conclusion

- Performance bound for RP and CCP_ε w.r.t. to SP
- Extend CCP_ε feasibility results to a class of non-convex programs

Outlook

- Computation of tail-bound function $h(\cdot)$
- Adaptive formulation
- Approximation method for infinite-dimensional LPs ($\rightarrow$ soon on arXiv)

Acknowlegements

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References:

- Mohajerin Esfahani, Sutter, and Lygeros, "Performance bounds for the scenario approach and an extension to a class of non-convex programs", *IEEE Transactions on Automatic Control*, vol. 60, no. 1, 2015