Averaging

Multiple estimates

Multiple experiments: $u_l(k), y_l(k), l = 1, \ldots, L,$ and $k = 0, \ldots, N - 1.$

Multiple estimates (ETFE):

$$\hat{G}_l(e^{j\omega_n}) = \frac{Y_l(e^{j\omega_n})}{U_l(e^{j\omega_n})}$$  \hspace{1cm} \text{(drop the $N$ from the $Y_N(e^{j\omega_n})$ notation)}

Averaging to improve the estimate.

$$\hat{G}(e^{j\omega_n}) = \sum_{l=1}^{L} \alpha_l \hat{G}_l(e^{j\omega_n}), \quad \text{with} \quad \sum_{l=1}^{L} \alpha_l = 1.$$  

How to choose $\alpha_l$?

The “average” is calculated with $\alpha_l = 1/L.$
Averaging

Optimal weighted average

If the variance of $\hat{G}_l(e^{j\omega_n})$ is $\sigma_l(e^{j\omega_n})$ (with uncorrelated errors and identical means), then,

$$\text{variance} \left( \hat{G}(e^{j\omega_n}) \right) = \text{variance} \left( \sum_{l=1}^{L} \alpha_l(e^{j\omega_n}) \hat{G}_l(e^{j\omega_n}) \right) = \sum_{l=1}^{L} \alpha_l^2 \sigma_l(e^{j\omega_n})$$

This is minimized by

$$\alpha_l(e^{j\omega_n}) = \frac{1/\sigma_l(e^{j\omega_n})}{\sum_{l=1}^{L} 1/\sigma_l(e^{j\omega_n})}.$$ 

If variance $\left( \hat{G}_l(e^{j\omega_n}) \right) = \frac{\phi_v(e^{j\omega_n})}{\frac{1}{N} |U_l(e^{j\omega_n})|^2}$ then

$$\alpha_l(e^{j\omega_n}) = \frac{|U_l(e^{j\omega_n})|^2}{\sum_{l=1}^{L} |U_l(e^{j\omega_n})|^2}.$$

Variance reduction

Best result: $|U_l(e^{j\omega_n})|$ is the same for all $l = 1, \ldots, L$.

This gives,

$$\text{variance} \left( \hat{G}(e^{j\omega_n}) \right) = \frac{\text{variance} \left( \hat{G}_l(e^{j\omega_n}) \right)}{L}.$$ 

If the estimates are biased we will not get as much variance reduction.

(The method of splitting the data record and averaging the periodograms is attributed to Bartlett (1948, 1950).)
Averaging example

Splitting the data record: \( N = 24, L = 3 \),

Estimates: \( \hat{G}_l(e^{j\omega_n}), i = 1, 2, 3 \) and the weighted average, \( \hat{G}(e^{j\omega_n}) \).
Averaging with periodic signals

Splitting the data record: $N = 24$, $L = 3$, with $u(k)$ periodic.

Estimates: $\hat{G}_l(e^{j\omega})$ and $\hat{G}(e^{j\omega}) = \left(\hat{G}_2(e^{j\omega}) + \hat{G}_3(e^{j\omega})\right)/2$

$|G_l(e^{j\omega})|$, $l = 2,..,L$ and $|G(e^{j\omega})|$
Smoothing transfer function estimates

What if we have no option of running periodic input experiments?

- The system will not tolerate periodic inputs; or
- The data has already been taken and given to us.
- More data just gives more frequencies (with the same error variance).

Exploit the assumed smoothness of the underlying system

\[ G(e^{j\omega}) \] is assumed to be a low order (smooth) system.

\[ E \left\{ (G(e^{j\omega}) - \hat{G}_N(e^{j\omega_n}))(G(e^{j\omega}) - \hat{G}_N(e^{j\omega_s})) \right\} \longrightarrow 0 \quad (n \neq s), \] (asymptotically at least).

Smoothing the ETFE

Smooth transfer function assumption

Assume the true system to be close to constant for a range of frequencies.

\[ G(e^{j\omega_n+l}) \approx G(e^{j\omega_n}) \quad \text{for } l = 0, \pm 1, \pm 2, \ldots, L. \]

Smooth minimum variance estimate

The minimum variance smoothed estimate is,

\[ \tilde{G}_N(e^{j\omega_n}) = \frac{\sum_{l=-L}^{L} \alpha_l \hat{G}_N(e^{j\omega_n+l})}{\sum_{l=-L}^{L} \alpha_l}, \quad \alpha_l = \frac{1}{N} \left| U_N(e^{j\omega_n+l}) \right|^2 / \phi_v(e^{j\omega_n+l}). \]
If $N$ is large (many closely spaced frequencies), the summation can be approximated by an integral,

$$
\hat{G}_N(e^{j\omega_n}) = \frac{1}{L} \sum_{l=-L}^{L} \alpha_l \hat{G}_N(e^{j\omega_n + l})
$$

$$
= \frac{1}{\omega_n-l} \int_{\omega_n-l}^{\omega_n+l} \alpha(\xi) \hat{G}_N(e^{j\xi}) d\xi
$$

with

$$
\alpha(\xi) = \frac{1}{N} \frac{|U_N(e^{j\xi})|^2}{\phi_v(e^{j\xi})}.
$$

The $1/2\pi$ scaling will make it easier to derive time-domain windows later.
Smoothing the ETFE

Assumptions on $\phi_v(e^{j\omega})$

Assume $\phi_v(e^{j\omega})$ is also a smooth (and flat) function of frequency.

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} W_\gamma(\xi - \omega_n) \left| \frac{1}{\phi_v(e^{j\xi})} - \frac{1}{\phi_v(e^{j\omega_n})} \right| d\xi \approx 0.$$  

Then use,

$$\alpha(\xi) = \frac{1}{N} \left| U_N(e^{j\xi}) \right|^2 \frac{1}{\phi_v(e^{j\omega_n})},$$

to get,

$$\tilde{G}_N(e^{j\omega_n}) = \frac{\frac{1}{2\pi} \int_{-\pi}^{\pi} W_\gamma(\xi - \omega_n) \frac{1}{N} \left| U_N(e^{j\xi}) \right|^2 \hat{G}_N(e^{j\xi}) d\xi}{\frac{1}{2\pi} \int_{-\pi}^{\pi} W_\gamma(\xi - \omega_n) \frac{1}{N} \left| U_N(e^{j\xi}) \right|^2 d\xi}.$$  

Weighting functions

Window characteristics: width (specified by $\gamma$ parameter)

The wider the window (i.e. decreasing $\gamma$) ...

- the more frequencies included in the estimate,
- the smoother the result,
- the lower the noise induced variance,
- the higher the bias.
Weighting functions

Typical window (Hann) as a function of the width parameter, $\gamma$.

\[
W_\gamma(e^{j\omega n})
\]

Discrete frequency: $\omega$

Weighting functions

Window characteristics: shape

Some common choices:

**Bartlett**:

\[
W_\gamma(\omega) = \frac{1}{\gamma} \left( \frac{\sin \frac{\gamma \omega}{2}}{\sin \frac{\omega}{2}} \right)^2
\]

**Hann**:

\[
W_\gamma(\omega) = \frac{1}{2} D_\gamma(\omega) + \frac{1}{4} D_\gamma(\omega - \pi/\gamma) + \frac{1}{4} D_\gamma(\omega + \pi/\gamma)
\]

where

\[
D_\gamma(\omega) = \frac{\sin \omega(\gamma + 0.5)}{\sin \omega/2}
\]

Others include: Hamming, Parzen, Kaiser, ...

The differences are mostly in the leakage properties of the energy to adjacent frequencies. And the ability to resolve close frequency peaks.
Weighting functions

Example frequency domain windows.

\[ W_{\gamma}(e^{j\omega n}) \]

\( \gamma = 10 \)

Welch
Hann
Hamming
Bartlett

\[ W_{\gamma} e^{j\omega n} \]

Discrete frequency: \( \omega \)

ETFE smoothing example: MATLAB calculations

```matlab
U = fft(u); % calculate N point FFTs
Y = fft(y);
Gest = Y./U; % ETFE estimate
Gs = 0*Gest; % smoothed estimate

omega = (2*pi/N)*[0:N-1]'; % frequency grid
Wg = WHfdom(gamma,omega); % window
a = U.*conj(U); % variance weighting

for wn = 1:N,
    Wnorm = 0; % reset normalisation
    for xi = 1:N,
        widx = mod(xi-wn,N)+1; % wrap window index
        Gs(wn) = Gs(wn) + ... 
        Wg(widx) * Gest(xi) * a(xi);
        Wnorm = Wnorm + Wg(widx) * a(xi);
    end
    Gs(wn) = Gs(wn)/Wnorm; % weight normalisation
end
```
Window properties

Properties and characteristic values of window functions

\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} W_\gamma(\xi) d\xi = 1 \quad \text{(Normalised)} \]

\[ \int_{-\pi}^{\pi} \xi W_\gamma(\xi) d\xi = 0 \quad \text{("Even" sort of)} \]

\[ M(\gamma) := \int_{-\pi}^{\pi} \xi^2 W_\gamma(\xi) d\xi \]

\[ \bar{W}(\gamma) := 2\pi \int_{-\pi}^{\pi} W_\gamma^2(\xi) d\xi \]

Bartlett: \[ M(\gamma) = \frac{2.78}{\gamma}, \quad \bar{W}(\gamma) \approx 0.67\gamma \quad \text{(for } \gamma > 5) \]

Hamming: \[ M(\gamma) = \frac{\pi^2}{2\gamma^2}, \quad \bar{W}(\gamma) \approx 0.75\gamma \quad \text{(for } \gamma > 5) \]

Smoothed estimate properties

Asymptotic bias properties

\[ E \left\{ \hat{G}(e^{j\omega_n}) - G(e^{j\omega_n}) \right\} = M(\gamma) \left( \frac{1}{2} G''(e^{j\omega_n}) + G'(e^{j\omega_n}) \frac{\phi'_u(e^{j\omega_n})}{\phi_u(e^{j\omega_n})} \right) \]

\[ + O(C_3(\gamma)) \quad \text{as } \gamma \to \infty \]

\[ + O\left( \frac{1}{\sqrt{N}} \right) \quad \text{as } N \to \infty \]

Increasing \( \gamma \):

- makes the frequency window narrower;
- averages over fewer frequency values;
- makes \( M(\gamma) \) smaller; and
- reduces the bias of the smoothed estimate, \( \hat{G}(e^{j\omega_n}) \).
Asymptotic variance properties

\[
E \left\{ \left( \hat{G}(e^{i\omega_n}) - E \{ \tilde{G}(e^{i\omega_n}) \} \right)^2 \right\} = \frac{1}{N} \hat{W}(\gamma) \frac{\phi_u(e^{i\omega_n})}{\phi_u(e^{i\omega_n})} + o\left( \frac{\hat{W}(\gamma)}{N} \right) \xrightarrow{N \to \infty} 0 \text{ as } \gamma \to \infty
\]

Increasing \( \gamma \):
- makes the frequency window narrower;
- averages over fewer frequency values;
- makes \( \hat{W}(\gamma) \) gets larger; and
- increases the variance of the smoothed estimate, \( \tilde{G}(e^{i\omega_n}) \).

Asymptotic MSE properties

\[
E \left\{ \left| \hat{G}(e^{i\omega_n}) - G(e^{i\omega_n}) \right|^2 \right\} \approx M^2(\gamma) |R(e^{i\omega_n})|^2 + \frac{1}{N} \hat{W}(\gamma) \frac{\phi_u(e^{i\omega_n})}{\phi_u(e^{i\omega_n})},
\]

where \( R(e^{i\omega_n}) = \frac{1}{2} G''(e^{i\omega_n}) + G'(e^{i\omega_n}) \frac{\phi'_u(e^{i\omega_n})}{\phi_u(e^{i\omega_n})} \)

If \( M(\gamma) = M/\gamma^2 \) and \( \hat{W}(\gamma) = \hat{W} \gamma \) then MSE is minimised by,

\[
\gamma_{\text{optimal}} = \left( \frac{4M^2 |R(e^{i\omega_n})|^2 \phi_u(e^{i\omega_n})}{\hat{W} \phi_u(e^{i\omega_n})} \right)^{1/5} N^{1/5}.
\]

and

MSE at \( \gamma_{\text{optimal}} \approx CN^{-4/5} \)
Windowing and ETFE smoothing

