Residual spectrum

Estimating $\phi_v(e^{j\omega_n})$

\[ v(k) = y(k) - G(e^{j\omega})u(k), \]

In the frequency domain,

\[ V_N(e^{j\omega_n}) = Y_N(e^{j\omega_n}) - G(e^{j\omega_n})U_N(e^{j\omega_n}) \]
\[ \approx Y_N(e^{j\omega_n}) - \tilde{G}(e^{j\omega_n})U_N(e^{j\omega_n}) \]

To smooth this as an estimate for $\phi_v(e^{j\omega_n})$,

\[ \tilde{\phi}_v(e^{j\omega_n}) \approx \frac{1}{N} \left| V_N(e^{j\omega_n}) \right|^2 \]
\[ \approx \frac{1}{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} W_\gamma(\xi - \omega_n) \left| Y_N(e^{j\omega_n}) - \tilde{G}(e^{j\omega_n})U_N(e^{j\omega_n}) \right|^2 d\xi \]
Residual spectrum

\[ \tilde{\phi}_u(e^{j\omega_n}) \approx \frac{1}{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} W_\gamma(\xi - \omega_n) \left| Y_N(e^{j\omega_n}) - \tilde{G}(e^{j\omega_n})U_N(e^{j\omega_n}) \right|^2 d\xi \]

\[ \approx \frac{1}{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} W_\gamma(\xi - \omega_n) \left| Y_N(e^{j\omega_n}) \right|^2 d\xi \]

\[ + \frac{1}{N} \left| \tilde{G}(e^{j\omega_n}) \right|^2 \frac{1}{2\pi} \int_{-\pi}^{\pi} W_\gamma(\xi - \omega_n) \left| U_N(e^{j\omega_n}) \right|^2 d\xi \]

\[ - 2 \text{real} \left\{ \frac{1}{N} \tilde{G}(e^{j\omega_n}) \frac{1}{2\pi} \int_{-\pi}^{\pi} W_\gamma(\xi - \omega_n) U_N(e^{j\omega_n}) Y_N^*(e^{j\omega_n}) d\xi \right\} \]

\[ \approx \tilde{\phi}_y(e^{j\omega_n}) + \left| \frac{\tilde{\phi}_{yu}(e^{j\omega_n})}{\tilde{\phi}_u(e^{j\omega_n})} \right|^2 \tilde{\phi}_u(e^{j\omega_n}) - 2 \text{real} \left\{ \frac{\tilde{\phi}_{yu}(e^{j\omega_n})}{\tilde{\phi}_u(e^{j\omega_n})} \tilde{\phi}_{yu}(e^{j\omega_n}) \right\} \]

\[ \approx \tilde{\phi}_y(e^{j\omega_n}) - \left| \frac{\tilde{\phi}_{yu}(e^{j\omega_n})}{\tilde{\phi}_u(e^{j\omega_n})} \right|^2. \]

Coherency spectrum

How much energy is accounted for by the model? How much by noise?

\[ \hat{\kappa}_{yu}(e^{j\omega_n}) = \sqrt{\frac{\left| \tilde{\phi}_{yu}(e^{j\omega_n}) \right|^2}{\tilde{\phi}_y(e^{j\omega_n})\tilde{\phi}_u(e^{j\omega_n})}} \]

From the previous estimate of the disturbance spectrum,

\[ \hat{\phi}_v(e^{j\omega_n}) = \hat{\phi}_y(e^{j\omega_n}) \left( 1 - \left( \hat{\kappa}_{yu}(e^{j\omega_n}) \right)^2 \right). \]

If, at a frequency \( \omega_n \), all of the energy in the output is due to the model, then \( \hat{\kappa}_{yu}(e^{j\omega_n}) = 1. \)

This can be used as measure of effectiveness of the modeling at a particular frequency.

Theoretically, \( 0 \leq \hat{\kappa}_{yu}(e^{j\omega_n}) \leq 1. \)
Time-domain windowing

Windowed periodograms

Time-domain windows can also be applied directly to the data.

\[ U_N(e^{j\omega n}) = \sum_{k=0}^{N-1} w_\gamma(k - N/2)u(k)e^{-jk\omega n} \]

The \( N/2 \) shift moves the window peak to the center of the data record.

We would typically choose \( \gamma = N/2 \) to use all the data.

This is not the same as applying a time-domain window to \( \hat{R}_u(\tau) \).

Depending on the scaling in \( w_\gamma(k) \), this changes the energy in \( U_N(e^{j\omega n}) \).

Example: Bartlett window:
Time-domain windowing

Frequency domain result:

Periodogram — scaling is required for estimating spectra

\[
\text{Periodogram} = \frac{1}{NE_{\omega, \gamma}} \left| U_N(e^{j\omega_n}) \right|^2,
\]

where,

\[
E_{\omega, \gamma} = \frac{1}{N} \sum_{k=0}^{N-1} |w_\gamma(k)|^2.
\]
Time-domain windowing

Welch’s method

Split the data record into $L$ overlapping segments of length $N$.

\[ u_1(k) \quad u_3(k) \quad \cdots \]
\[ u_2(k) \quad \cdots \quad u_L(k) \]

\[ U_l(e^{j\omega_n}) = \sum_{k=0}^{N-1} w_\gamma(t - N/2)u_l(k)e^{-j\omega_n k} \]

\[ \tilde{\phi}_u(e^{j\omega_n}) = \frac{1}{NLE_{\omega_\gamma}} \sum_{l=1}^{L} \left| U_l(e^{j\omega_n}) \right|^2 \]

Advantages:

- Windowing can reduce transient response effects.
- Noise reduction from averaging and windowing.
- Variance error can be reduced.

Disadvantages:

- Windowing causes energy “leakage” to adjacent frequencies.
- Frequency resolution deteriorates.
- Bias error can be increased.
- Noise on $u_l(k)$ and $u_{l+1}(k)$ is not uncorrelated.
DFT analysis

Analysis of sinusoids

Signal:

\[ u(t) = \cos(\omega_1 t), \]

\[ \omega_1 = 1.2272 \text{ rad/sec}. \]

Sampling period: \( T = 0.1 \) seconds

Number of samples: \( N = 256. \)

This gives exactly 5 periods: \( \omega_1 = \frac{5 \times 2\pi}{TN}. \)

DFT analysis

Frequency domain (via DFT)

Magnitude and phase

Sinusoidal waveform: integer periods

Magnitude

Phase
DFT analysis

Analysis of sinusoids

Signal:

\[ u(t) = \cos(\omega_2 t), \]
\[ \omega_2 = 1.3499 \text{ rad/sec}. \]

Sampling period: \( T = 0.1 \) seconds

Number of samples: \( N = 256. \)

This gives 5.5 periods:

\[ \omega_2 = \frac{5.5 \times 2\pi}{TN}. \]
DFT analysis

Frequency domain (via DFT)

Magnitude and phase

Sinusoidal waveform: non-integer periods

Real and imaginary parts

real(U2) = -0.0625
DFT analysis

Frequency domain (via DFT)

Logarithmic scaling

Sinusoidal waveform: non-integer periods

Equivalent periodic signal

Equivalent signal: non-integer number of periods
DFT analysis

Relationship between Fourier Transform and DFT

Signal with a $Z$-domain representation:  
\[ X(z) = \sum_{k=-\infty}^{\infty} x(k)z^{-k} \]

Consider sampling this on the unit circle:

\[ X(e^{j\omega}) = X(z) \big|_{z=e^{j\omega}} = \sum_{k=-\infty}^{\infty} x(k)e^{-j\omega k} \]

What periodic signal, $\tilde{x}(k)$ corresponds to the same frequency samples?

\[ \tilde{x}(k) = \frac{1}{N} \sum_{n=0}^{N-1} X(e^{j\omega_n})e^{j\omega_n k} = \sum_{r=-\infty}^{\infty} x(k+rN). \]

So, if $x(k)$ is finite (length $N$), its DFT corresponds to the Fourier Series of a periodic extension of $x(k)$.

Windowing

Time-domain windowing

Hamming and Cosine$^2$ windows in the time domain:
Windowing

Time-domain windowing

Windowed signals

![Graph showing windowed signals with different windows: Hamming and Cosine](image)

Windowing effects

Frequency domain effects: (magnitude detail)

![Graph showing frequency domain effects with sinusoidal waveform](image)
Window details

Rectangular \( w(k) = 1 \)
Hamming \( w(k) = 0.08 + 0.46(1 - \cos(2\pi k/N)) \)
Cosine\(^2\) \( w(k) = 0.5(1 - \cos(2\pi k/N))^2 \)

Window characteristics: \( x \) denotes sample points

Window mainlobe details

Rectangular \( w(k) = 1 \)
Hamming \( w(k) = 0.08 + 0.46(1 - \cos(2\pi k/N)) \)
Cosine\(^2\) \( w(k) = 0.5(1 - \cos(2\pi k/N))^2 \)

Window characteristics: \( x \) denotes sample points
Window side lobe details

- Rectangular: \( w(k) = 1 \)
- Hamming: \( w(k) = 0.08 + 0.46(1 - \cos(2\pi k/N)) \)
- Cosine\(^2\): \( w(k) = 0.5(1 - \cos(2\pi k/N))^2 \)

Offsets

Suppose we have a dc offset on a measured signal.
Offsets

The offset has low frequency components that the window does not remove.

Drifts

\[ u(k) = \cos(\omega_1 k) + \alpha k \quad (\alpha \text{ unknown}) \]
Drifts

\[ u(k) = \cos(\omega_1 k) + \alpha k \quad (\alpha \text{ unknown}) \]

Windowing the signal removes the discontinuity at the ends.
Windowing reduces (but doesn’t eliminate) the effect of drifts.

\[ u_1 \]
\[ u_1 + \text{drift} \]
\[ \text{window: } u_1 + \text{drift} \]

Data preprocessing

**Detrending**

Preprocess the data via,

\[ \hat{u}(k) = u(k) - (\alpha k + \beta), \]

where, \(\alpha k + \beta\) is the best linear fit to \(u(k)\).

This will exactly remove offsets and drifts (for periodic \(u(k)\)).

The MATLAB command for this is `detrend(u)`.
Bibliography

**Residual spectra & coherency**

**Welch method**

**FFT analysis**